SIMPLE AND APPROXIMATELY OPTIMAL MECHANISMS DESIGN

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Mechanism design studies optimization problems with inputs from selfishly behaving agents. To a class of relatively simple questions, classical auction theory gives optimal solution which are often too complicated for practical use. This dissertation presents techniques and results on the design and analysis of simple and practical auctions that have provably approximately optimal performances. For single-parameter revenue optimization, we present the k-lookahead auction, which works for coorelated valuation distributions, and we discuss its consequences, which encompass a family of reserve-price-based auctions with approximately optimal revenue. For more general settings, we present the marginal revenue mechanisms as a framework for designing simple mechanisms in very general settings. Both frameworks are motivated with clear economic intuitions.

For social welfare maximization in combinatorial auctions, we present an equilibrium analysis for simultaneous item auctions. In particular, we show that when valuations exhibit no complements, the welfare loss in these auctions at Bayesian Nash equilibria is bounded by small constants.

BIOGRAPHICAL SKETCH

Hu Fu was born in Qingdao, China in 1984. Piano practice and performance occupied much of his childhood and has ever since been a nontrivial diversion. Hu won a gold medal in the Chinese National Biology Olympiad in 2002, and went on to the Department of Automation at Tsinghua University for his college education. There he won the prestigious Jiang Nanxiang Scholarship in 2004, and obtained his bachelor degree in 2007. Since then he has been pursuing his graduate study in computer science at Cornell University.

This dissertation is dedicated to my parents, and to the memory of my grandfather, Huizan Fu.

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It is most certain that the luckiest thing that happened to me in graduate school was to have found Bobby Kleinberg as my advisor. Not only did Bobby allure me to the wonderland of theoretical computer science, set me a model of academic brilliancy brimming with dazzling ideas, and provide crucial help with all my key career decisions, but also, in the true spirit of academic genealogy, he inculcated in me a taste for research and mathematics in general, an influence whose effect I expect to perceive for years to come. It was additional bliss to have Bobby as a genuine friend, to share and play music and to discuss all intellectual matters with him.

Among the colleagues I have collaborated with, Jason Hartline and Shahar Dobzinski have virtually come to assume the role of second advisors. I owe to them the inspiration of some of my best research outputs, and have repeatedly benefited from their wise advice.

I was born into a family whose intellectual tradition was disrupted for a generation by social upheavals. Fostering my intellectual growth therefore exacts additional sacrifice and patience from my parents, for which I am forever grateful. My grandparents, especially my late grandfather, sowed in me the scholarly seed. Under their and my parents' care, it budded through hardships not easily conceived in this country, and is now finally bearing some flowers. To think of the joy and pride this brings to them emboldens me before all obstacles in life.

The way I think about math was greatly changed and ripened by courses I took from Eugene Dynkin and Ken Brown. My enjoyment in graduate school was enriched by piano lessons taken from Xak Bjerken, and my intellectual horizon was tremendously broadened by my three years of residence in the Telluride House. Friendships I formed in the Computer Science Department and at the Telluride House, such as those with Thành Nguyen, Tuan Cao, Huijia Lin, Chenhao Tan, Hussam Abu-Libdeh, Judah Bellin, Kate Scheibel, Liz Solten, Sarah Asman, Jacob Krell and Srinath Reddy, are cherished in memory and will for sure be a lifelong treasure.

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CHAPTER 1
INTRODUCTION

Mechanism design studies optimization systems that take into account participants' private information and selfish behaviors. The past ten years have seen a fruitful marriage of this subfield of microeconomics with computer science, accelerated by the advent of the Internet and the ensuing business on it, such as display ad auctions run by search engine companies and large scale, dynamic auctions run by websites like Priceline or eBay. Algorithmic ideas and tools have been brought to bear on decades-old problems of economic theory, yielding new economic insights in addition to potential impacts on industries.

This thesis focuses on two aspects of this development. First, economic characterizations of optimal mechanisms beyond classic scenarios tend to be too complex for practical use. A theme of the chapters to follow to that, when the viewpoint of approximation is adopted, much more structure and insights can be revealed and applied. Second, and relatedly, the use of mechanisms in practice often favors simplicity over theoretical optimality, when the two design goals come into conflict. Simplicity here may refer to systems having succinct specifications, requiring less self-deliberation and private information disclosure by market participants, tolerating the lack of certain information on the mechanism implementer's part, and so on. This thesis presents research progress showing, on one hand, that the structure gained from adopting approximations often guides the design of simple mechanisms, and on the other hand, that mechanisms used in practice can often be analyzed and evaluated as approximations to optimal benchmarks.

Background. A mechanism involves the participation of strategic agents with private information. The agents' information, summarized as *types*, is unknown to us, while a decision on a social outcome needs to be made from interactions with them. In an auction, for example, a bidder's type may describe her valuations of the items being sold, and the social outcome is a decision to sell which items to which bidders at what prices. The mechanism designer devises rules of interaction and decision, so that certain goals, usually functions of the agents' types, can be optimized, while taking heed that the agents may behave strategically—their interests, expressed in *utilities*, are typically affected by the social outcome. A bedrock solution concept in mechanism design is *incentive compatibility* in *direct revelation* mechanisms. In a direct revelation mechanism, each agent reports her type to the mechanism; the mechanism is said to be *incentive compatible* if all agents of all types find it in their best interests to reveal information honestly. On this solution concept stand two Nobel-prize-winning works. One is the Vickrey-Clark-Groves (VCG) mechanism [18, 28, 54], a general procedure maximizing the social welfare (the sum of all participants' utilities). The other is Myerson's auction that optimizes the seller's expected revenue [43]. These two mechanisms bolster a large part of the study of mechanism design, but beyond very simple settings they either cannot be generalized or have too complex (economically or computationally or both) generalizations to be practical. A recurring theme of this thesis is to restore simplicity to these scenarios by way of adopting a viewpoint of approximations.

1.1 (Approximately) Optimal Auctions Beyond Myerson.

The optimal auction by Myerson maximizes the seller's expected revenue for bidders with linear utilities and independently drawn valuations. (An agent is said to have linear utility if her utility is equal to her value of the service or items she gets minus the payment she makes.) Notwithstanding its elegance, Myerson's auction has several limitations. First, unlike auctions commonly run in practice, where the outcome is typically transparently decided by comparisons among the bidders' bids above reserve prices, Myerson's auction is relatively complicated. One manifestation of this complexity is that a bidder with a lower bid can at times win over higher bidders. Second, when bidders' valuations are correlated, its revenue can be far from optimal [43]. Third, Myerson's optimal auction does not generalize to settings where bidders have multi-parameter types or non-linear utilities.

In Chapter 3 we investigate the first two limitations. A formally simple solution to the correlated distribution cases turns out to be instrumental with developing simple approximations in the independent value case. We start with the following simple *k-lookahead* auction, generalizing the lookahead auction proposed by Ronen [48]: conditioning on lower bids, run the optimal auction on the highest k bidders. We show the revenue of this auction to be at least $\frac{2k-1}{3k-1}$ of the optimal revenue extractable if we require that a truthful bidder never ends up with a negative utility. This result quantifies the importance of high bidders in terms of approximations. Lookahead auctions were originally developed for correlated valuations, but their analysis sheds light on reserve-price-based practical auctions under independent valuations. In the second part of Chapter 3, we present several such auctions with approximately optimal performance that are simpler than the optimal auctions in various aspects. We note that the latter results were known prior to this work and were the content of several papers [4, 21, 32], but the perspective of lookahead auctions allow great simplification of their analysis and allow us to present all of them rather compactly.

Myerson's requirement for single-parameter types and linear utilities significantly restricts the auction's applicability. For example, multi-item auctions generally require more than one parameter to describe bidders' valuations, and bidders that have budgets or are averse to risks cannot be modeled with linear utilities. Recently, a generic polynomial-time reduction was discovered [1, 11, 12, 13] from the problem of multi-agent optimal mechanism design to that of single-agent mechanism design. These algorithms are based on convex programming techniques, and produce complex procedures that are neither practical to run nor conducive to economic intuition. In Chapter 4, we present a framework for searching and analyzing simple and near-optimal mechanisms in these general environments. In a classical paper [9], Bulow and Roberts gave a simple interpretation of Myerson's optimal auction: if we see each bidder as a market wherein the makeup of customers is specified by the distribution of the bidder's types, then a monopoly price setter observes a shrinking fraction of buying customers as the price rises. When optimizing revenue with a limited supply over multiple markets, the monopolizer rations the supply across the markets by maximizing the sum of *marginal revenues*. Myerson's optimal auction turns out to be the same procedure, by mapping each type to the marginal revenue corresponding to that customer's critical buying price. This simple economics does not generalize to more complex settings, because there is no obvious mapping from a type to a marginal revenue, and because the mechanism designer has more freedom than simply setting prices in markets.[1] We develop techniques to connect both broken links, first by giving substance to *marginal revenue maximization* in general contexts, and second by showing that the procedure achieves *approximately* optimal revenue. Such mechanisms provide new insights on the economic structure of optimal mechanisms beyond Myerson's setting and have broad potential applications. One example we give is a simple, near-optimal auction for budgeted bidders in very general environments.

1.2 Towards More Realistic Mechanisms

Mechanisms used in reality are often not direct revelation incentive compatible but are simple procedures in which agents have room for strategic behaviors. Such mechanisms are popular for various reasons. For example, the communication between participants

[1] Such freedom happens not to render additional power in the single-parameter, linear utility case, which is the key to the relative straightforwardness of the analysis for these cases.

in such auctions can be significantly less than a full description of their types. The Generalized Second Price Auctions (GSP) run by search engines and the auctions on eBay are examples of such non-incentive-compatible auctions. As these auctions are often designed based on heuristics, it is of both theoretical and practical interest to analyze their performance at equilibrium reached by strategic participants.

As an example, we consider combinatorial auctions: m items are sold to n bidders, where each bidder has a valuation for every possible subset of items, and we would like to maximize the social welfare. A salient example of this is the auction held by the Federal Communications Commission (FCC) that sells spectrum to wireless communication companies. Instead of eliciting combinatorial preferences from participants, it is much more practical to simultaneously run first- or second-price auctions on the items separately. (In fact, such auctions are often used in practice; the auctions on eBay can be seen as an example after reasonable simplification.) Such simultaneous auctions form games where truthfully bidding one's valuation on each item can be a very poor strategy, and (Bayesian) Nash equilibrium is an appropriate concept with which to predict and evaluate the auctions' outcome. In Chapter 5 we show that, for simultaneous first-price (and second-price) auctions with complement-free valuations, the social welfare at any Bayesian Nash equilibrium is at least a half (and a quarter) of the optimal allocation.

1.3 Bibliographic Notes

The material in Section 3.2 is from joint work with Shahar Dobzinski and Bobby Kleinberg which appeared in [22], and the other parts of Chapter 3 are based on [26]. Chapter 4 is based on joint work with Saeed Alaei, Nima Haghpanah and Jason Hartline which will appear in [2]. Chapter 5 is based on joint work with Michal Feldman, Nick Gravin and Brendan Lucier, which appeared in [25].

CHAPTER 2

BACKGROUND MATERIAL

In this chapter we give basic definitions in mechanism design, introduce notations to be used throughout the rest of the thesis, and review a few classical results.

Single-Dimensional Mechanism Design For most of the thesis we will consider single-dimensional mechanism design problems. In Section 4.3 and Chapter 5 we will consider multi-dimensional settings. Notations and backgrounds will be provided therein.

In a single-dimensional auction environment, an auctioneer provides a service to a set N of n bidders, but with certain constraints on which subsets of bidders he could simultaneously serve. Formally, if we use $[n]$ to denote the set of bidders, then there is a set $\mathcal{I} \subseteq 2^{[n]}$, such that the auctioneer can serve a subset S of bidders simultaneously if and only if S is in $\mathcal{I}$. For example, in a single item auction, $\mathcal{I}$ consists of all singleton subsets of $[n]$. When $\mathcal{I}$ forms a matroid on $[n]$, we call the environment a *matroid setting*. Matroid settings encompass important types of markets, e.g. digital goods (when all subsets are feasible, i.e., $\mathcal{I} = 2^{[n]}$), k unit auctions (when $\mathcal{I}$ contains all subsets of size at most k, i.e., is the k-uniform matroid), and unit-demand bipartite matching markets (when $\mathcal{I}$ is a traversal matroid). We say the environment is *downward closed* if $T \in \mathcal{I}$ implies $S \in \mathcal{I}, \forall S \subseteq T$.

Valuations and Their Distributions Each bidder i has a private valuation v_i for being served. We will assume that v_i is drawn from a distribution whose cumulative distribution function is F_i, i.e., $F_i(v) = \Pr[v_i \leq v]$. The corresponding density function is denoted as f_i. When the valuations of different bidders are independent random variables, we say they are independent bidders; otherwise they are *correlated*. The joint distribution of all bidders' values is denoted by F. Except for the part on the revenue guarantee of lookahead auctions (Section 3.2 and relavant theorems in Section 3.3) , we focus on auctions with independent bidders in this thesis. When all bidders' valuations are identically independently distributed (i.i.d.), we say they are symmetric; otherwise they are asymmetric. A standard shortened notation for $(v_1, \ldots, v_{i-1}, v'_i, v_{i+1}, \ldots, v_n)$ is (v'_i, v_{-i}).

Auctions and Incentive Compatibility An *auction* is a pair of vector functions (x, p) defined on tuples of valutions. At the valuations $(v_1, \ldots, v_n)$, the *allocation* function, $x_i(v_1, \ldots, v_n)$, gives the probability with which bidder i receives the service, and $p_i(v_1, \ldots, v_n)$, the *payment* function, denotes the expected payment bidder i makes to the auctioneer. Bidder i is said to have *linear utility*, or, equivalently, *risk averse*, if his utility in participating the auction (x, p), given the other bidders' valuations v_{-i}, is $v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})$. We will consider risk neutral bidders throughout the thesis except in Section 4.3.

Since bidders' valuations are private and their utility is affected by the auction outcome, which in turn depends on the bidders' behaviors, the bidders may misreport their valuations to gain advantage. In order to incentivize truthful behavior, we study auctions that are *incentive compatible*. An auction is called *dominant strategy incentive compatible* (DSIC) or, equivalently, *ex post incentive compatible*, if, for all $(v_1, \ldots, v_n)$, all i and all v'_i,

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i x(v'_i, v_{-i}) - p_i(v'_i, v_{-i}).$$

An auction is called *Bayesian incentive compatible* (BIC) if, for all i, v_i and v'_i,

$$\mathbf{E}_{v_{-i} \sim F_{-i \mid v_i}} \left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \right] \geq \mathbf{E}_{v_{-i} \sim F_{-i \mid v_i}} \left[v_i x_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i}) \right],$$

where $F_{-i \mid v_i}$ is the conditional distribution of v_{-i} given v_i. When the bidders are independent, this is just $\prod_{j \neq i} F_j$.

An auction is said to be *ex post individually rational* (ex post IR) if, for all $(v_1, \ldots, v_n)$

and all i,

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0.$$

Similarly to the definition of BIC, we say that a mechanism is *interim individually rational* (interim IR), if for all i and v_i,

$$\mathbf{E}_{v_{-i} \sim F_{-i \mid v_i}} \left[v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \right] \geq 0.$$

In this thesis we will focus on incentive compatible auctions in Chapter 3 and Chapter 4, where we will use the terms *bids* and *valuations* interchangeably. In Chapter 5 we will consider two auctions that are not incentive compatible.

The *social welfare* for serving a set S of bidders is $\sum_{i \in S} v_i$. The expected revenue made by an auction is $\mathbf{E}_{(v_1, \ldots, v_n) \sim F}[\sum_i p_i(v_1, \ldots, v_n)]$.

Characterization of BIC Mechanisms and Optimal Auctions In a seminal work, Myerson [43] gives a characterization of all BIC mechanisms in single-dimensional environments with risk-neutral bidders, and also derives the revenue-optimal auction. These results are summarized in the following theorem, where part 4 is a corollary on single-item auctions by applying the other parts:

Theorem 2.0.1 (43)**.** *For risk-neutral bidders with valuations drawn independently and identically from F,*

1. *(monotonicity) The allocation rule $x(v)$ for each agent is monotone non-decreasing in v.*
2. *(payment identity) The payment rule satisfies $p(v) = vx(v) - \int_0^v x(z)\mathrm{d}z$.*
3. *(virtual value) The ex ante expected payment of an agent is $\mathbf{E}_v[p(v)] = \mathbf{E}_v[\varphi(v)x(v)]$ where $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ is the* virtual value *for value v.*
4. *(optimality) In a single-item auction, when the distribution F is* regular*, i.e., $\varphi(v)$ is monotone, the second-price auction with reserve $\varphi^{-1}(0)$ is revenue-optimal.*

Revenue Curves and Monopoly Reserves One way to interpret Myerson [43]'s characterization is by way of *revenue curves*. We will repeated make use of this; in fact, Chapter 4 is virtually devoted to a reinterpretation and generalization of this important perspective.

Consider selling to a single bidder i by setting a take-it-or-leave-it price. For each price p, the probability that the bidder takes the price is $q = 1 - F(p)$, and the expected revenue is $p(1 - F(p))$. Conversely, for each probability q of selling, the expected revenue from setting the corresponding price is $q \cdot F^{-1}(1 - q)$. The *revenue curve* for a valuation distribution is the plot of this expected revenue against $q \in [0, 1]$. The price corresponding to the highest point of the revenue curve is called the *monopoly reserve price*, denoted by r^{mon}. [1] A consequence of Theorem 2.0.1 is that the optimal auction for selling to a single bidder is to set the monopoly reserve as the take-it-or-leave-it price.

Importantly, the derivative of the revenue curve at a quantile q is equal to the virtual value at the corresponding quantile $\varphi(F^{-1}(q))$ (we will explain this in more detail in Chapter 4). Therefore, the condition that $\varphi(v)$ is a monotone increasing function (i.e., that F is a *regular* distribution), is equivalent to the condition that the revenue curve should be concave.

VCG auctions with eager and lazy reserve prices Another classic auction which lies at the foundation of much work on mechanism design is the VCG auction [54, 18, 28]. In the VCG auction, the winners are the set of bidders that achieves the maximum sum of valuations over all feasible sets, and each of them pays the critical valuation below which he would drop from the winning set. Formally, the set W of winners is $\operatorname{argmax}_{S \in \mathcal{I}} \sum_{i \in S} v_i$, and the payment made by a bidder in W is $p_i^{\text{VCG}} = \inf\{v_i \mid i \in \operatorname{argmax}_{S \in \mathcal{I}} \sum_{j \in S} v_j\}$. This payment p_i^{VCG} is called the *VCG payment* or *VCG threshold* for bidder i. It is a

[1] When there are multiple prices that maximize the revenue, we always pick the highest price among them (corresponding to the lowest q) as the monopoly reserve price, throughout the thesis.

quantity defined solely by the valuations of other bidders and the feasibility system $\mathcal{I}$, and therefore is well defined for all bidders, including those who lose the auction. For each bidder, the VCG threshold is the value above which he will win the auction, and below which he loses. Another interpretation of the VCG payment that we will use in Section 3.3.3 is that it represents the externality of a winner imposed on the rest of the bidders, i.e., for a winner i, p_i^{VCG} is the decrease in welfare among all other bidders caused by bidder i's presence.

Two important facts about VCG auctions are that they are dominant strategy incentive compatible and they maximize the social welfare. A special case of the VCG auction is the second-price single-item auction, where the highest bidder wins the item and pays the second highest bid.

CHAPTER 3

LOOKAHEAD AUCTIONS AND THEIR CONSEQUENCES

In this chapter we start by considering the problem of revenue maximization in a single-item auction with bidders having correlated valuations, under the constraints of dominant strategy incentive compatibility and ex post individual rationality. A simple and elegant solution to this problem is the lookahead auction proposed by Ronen [48], who showed that it gives a 2-approximation to the optimal revenue. In Section 3.2, we present a generalization, the k-lookahead auctions, which we show gives a $\frac{3k-1}{2k-1}$-approximation to the optimal revenue. Then in Section 3.3 we use the idea of the lookahead auction to study reserve-price-based auctions, which are much simpler and more commonly used than Myerson's optimal auctions. Using lookahead auctions, we give simple proofs for the approximation guarantees of several such auctions.

3.1 Preliminary

For bidders having correlated valuations, Crémer and McLean [19] showed that, for distributions that satisfy a certain "correlation condition", one can extract *full social surplus* by a DSIC but interim IR auction. In such an auction, the expected revenue is equal to the expected optimal social welfare, the maximum revenue one can hope for. However, it is crucial that the mechanism is interim IR only, and not ex post IR, as the mechanism can sometimes, depending on the value profiles, even charge a large amount from a bidder who does not win the item. When this is undesirable, as for example when a bidder has a budget that prevents her from paying more than her value of an outcome, ex post IR is a more realistic requirement. While the optimal auction in this case for two bidders is completely solved [45, 22], it is known that computing the optimal deterministic auction for at least three bidders is NP-hard [45], and the only known algorithm for computing the optimal randomized mechanism is LP-based, with a running time exponential in n in the worst case [22]. In Section 3.2 we present an ex post IR auction that approximates the optimal revenue of any randomized mechanism.

The auction is a generalization of the following simple auction proposed by Ronen [48] which he called the *lookahead auction*: choose the highest bidder as the tentative winner; run an optimal auction for the highest bidder on a conditional distribution of his valuation, where the conditioning is on all other bidders' valuations and the fact that this tentative winner's valuation is above all others'. It is not hard to see that the lookahead auction is dominant strategy incentive compabible.

Theorem 3.1.1 (48)**.** *The lookahead auction gives at least half of the optimal revenue in a single item auction, even for bidders with correlated valuations.*

Ronen's proof is elementary, and it is instructive to repeat it here for the purpose of our generalization in Theorem 3.3.8. The revenue from any auction consists of two parts: H, the revenue extracted from the highest bidder; and L, the revenue extracted from the rest of the bidders. Since lookahead always runs the optimal auction for the highest bidder with all the information present, its revenue is at least H. One way to run the auction for the highest bidder is simply to charge him the second highest valuation, which is an upper bound on L. Therefore the lookahead auction's revenue is no less than both H and L, and hence gets at least half of the optimal revenue.

3.2 The k-lookahead Auction

In this section we generalize the lookahead auction and analyze the k-lookahead auction defined as follows. Find the k bidders with the highest values, and denote this set of bidders by K. Run the revenue-maximizing truthful auction for K conditioned on the values of bidders in $N \setminus K$ and the fact that all of the bidders in K have valuations than those in $N \setminus K$. Notice that the auction for K can either be the optimal truthful-in-expectation mechanism or the optimal deterministic mechanism.

Theorem 3.2.1. *The approximation ratio of the k-lookahead auction is at least* $\frac{3k-1}{2k-1}$*. In particular, for* $k = 2$ *the approximation ratio is at least as good as* $\frac{5}{3}$*, and the approximation ratio is bounded above by* $\frac{3}{2}$ *as k tends to infinity.*

3.2.1 Analysis of the k-Lookahead Auction

Denote the original distribution by $\mathcal{D}$, and denote by $\mathcal{D}_K$ the conditional distribution of the values of the bidders in K given the values of bidders in $N \setminus K$ and the fact that bidders in K all have higher valuations than those in $N \setminus K$. We let v_{k+1} denote the the value of the $(k+1)^{\text{th}}$-highest bidder. We show that one of the following three families of auctions provides a good approximation ratio. The k-lookahead auction obviously provides at least as much expected revenue, and the theorem follows. The auctions are defined for $k \geq 2$. The second and third auctions depend on a parameter $t \geq 1$, to be specified later.

1. **k-Highest Auction:** Run the optimal auction. If one of the bidders in $N \setminus K$ is assigned the item in the optimal auction, no bidder is assigned the item and no one is charged anything. If one of the bidders in K is assigned the item in the revenue-maximizing auction then assign him the item and charge him as in the revenue-maximizing auction.

2. **t-Fixed Price Auction:** Select one bidder ("the reserve bidder") from K uniformly at random, denote this bidder by i. If any of the bidders in $K \setminus \{i\}$ has value above $t \cdot v_{k+1}$ then he receives the item and pays $t \cdot v_{k+1}$. If there are several such bidders, break ties arbitrarily. Otherwise, the reserve bidder gets the item and pays v_{k+1}.

3. **t-Pivot Auction:** Select one bidder ("the pivot") from K uniformly at random, and denote this bidder by i. If any of the bidders of in $K \setminus \{i\}$ has value above $t \cdot v_{k+1}$ then

run the revenue maximizing auction for bidders in K, conditioned on the values of bidders in $N \setminus K$ and the fact that all bidders in K have higher valuations than those in $N \setminus K$. Otherwise the pivot bidder gets the item and pays v_{k+1}.

It is straightforward to see that the k-Highest Auction and the t-Fixed Price Auction are truthful and individually rational[1]. To see that the t-Pivot Auction is truthful we observe that this auction is monotone: the only non-straightforward case to check is when a bidder j other than i raises his value and forces the mechanism to run the optimal auction. However, in this case bidder j was not allocated the item before raising his value, so monotonicity is preserved.

Proof of Theorem 3.2.1. Throughout this proof, we condition on a fixed valuation profile on $N \setminus K$. We show that the conditional expected revenue from the k-lookahead auction approximates the conditional expected revenue from the optimal auction, and therefore its expected revenue regardless of the conditioning is still a $\frac{3k-1}{2k-1}$-approximation.

Let l be the event where no bidder in K has value at least $t \cdot v_{k+1}$, and let $\bar{l}$ be the complement of this event. We partition the expected revenue of the optimal auction:

- let L_l be the expected revenue from bidders in $N \setminus K$ from instances where event l occurs.
- let $L_{\bar{l}}$ be the expected revenue from bidders in $N \setminus K$ from instances where event $\bar{l}$ occurs.
- Let M be the expected revenue from bidders in K from instances where event l occurs.
- Let H be the expected revenue from bidders in K from instances where event $\bar{l}$ occurs.

[1]If the optimal auctions used by the k-Highest Auction and the t-Pivot Auction are deterministic or universally truthful, then all three auctions are universally truthful, as is their convex combination. In this case our proof shows that there is a deterministic auction on the k highest bidders that achieves a $\left(\frac{3k-1}{2k-1}\right)$-approximation to the deterministic optimal auction.

Observe that the expected revenue of the optimal auction is $L_l + L_{\bar{l}} + H + M$. We continue by proving several lemmas.

Lemma 3.2.2. *The expected revenue of the k-Highest Auction is $M + H$.*

Proof. By definition the auction extracts exactly the same revenue as the optimal auction from bidders in K and no revenue from bidders in $N \setminus K$. The lemma follows. □

Lemma 3.2.3. *The expected revenue of the t-Fixed Price Auction is at least $L_l + \frac{M}{t} + \frac{k-1}{k} \cdot t \cdot L_{\bar{l}} + \frac{1}{k} \cdot L_{\bar{l}}$.*

Proof. First, notice that the revenue of the t-Fixed Price Auction in every instance is at least v_{k+1} (either the reserve bidder is allocated the item and pays v_{k+1} or the auction sells the item at a higher price). Suppose that event l occurs. This case contributes $L_l + M$ to the expected revenue of the optimal auction. Observe that, if l occurs, in any instance where the optimal auction sells the item to bidders in $N \setminus K$, its revenue is at most v_{k+1} (the price for a sold item is at most the value of the bidder), and that in any instance the optimal auction sells the item to bidders in $N \setminus K$ the revenue is at most $t \cdot v_{k+1}$. Thus, the instances where event l occurs contribute $L_l + \frac{M}{t}$ to the expected revenue of the t-Fixed Price Auction.

Suppose now that event $\bar{l}$ occurs. Thus, there exists some bidder b with $v_b > t \cdot v_{k+1}$. With probability exactly $\frac{k-1}{k}$, b is not the reserve bidder and in this case the revenue of the auction is $t \cdot v_{k+1}$. With probability $\frac{1}{k}$ we have that b is the reserve bidder and the revenue of the auction is at least v_{k+1}. In particular, for every instance where the optimal auction sells the item to bidders in $N \setminus K$ (at a price of at most v_{k+1}) the t-Fixed Price Auction has an expected revenue of at least $\frac{k-1}{k} \cdot t \cdot L_{\bar{l}} + \frac{1}{k} \cdot L_{\bar{l}}$. Together with the contribution from instances where event l occurs we have that the expected revenue of the auction is at least $L_l + \frac{M}{t} + \frac{k-1}{k} \cdot t \cdot L_{\bar{l}} + \frac{1}{k} \cdot L_{\bar{l}}$. □

Lemma 3.2.4. *The expected revenue of the t-Pivot Auction is at least $L_l + \frac{M}{t} + \frac{k-1}{k} H + \frac{1}{k} L_{\bar{l}}$.*

Proof. Suppose that event l occurs. The revenue of the t-Pivot Auction in every instance where l occurs is v_{k+1}. The expected revenue of the optimal auction in this case is $L_l + M$, and similarly to the the analysis of the t-Fixed Price Auction the expected contribution to the revenue from instances where event l occurs is $L_l + \frac{M}{t}$.

Suppose that event $\bar{l}$ occurs. Thus, there exists some bidder b with $v_b > t \cdot v_{k+1}$. With probability exactly $\frac{k-1}{k}$, b is not the reserve bidder and in this case the revenue of the auction is at least H. With probability $\frac{1}{k}$ we have that b is the reserve bidder and the revenue of the auction is at least $\frac{1}{k} \cdot v_{k+1}$. Again, similarly to the analysis of the t-Fixed Price Auction the expected contribution to the revenue when the event $\bar{l}$ occurs is $\frac{k-1}{k}H + \frac{1}{k}L_{\bar{l}}$. Overall, the expected revenue of the auction is at least $L_l + \frac{M}{t} + \frac{k-1}{k}H + \frac{1}{k}L_{\bar{l}}$. □

Next we need some definitions. Conditioned on the values of bidders in $N \setminus K$, let OPT be the revenue of the revenue-maximizing auction, R_h be the revenue of the k-Highest Auction, R_f be the expected revenue of the t-Fixed Price Auction, R_p the expected revenue of the t-Pivot Auction and $R = \max(R_f, R_p)$. In addition, for the rest of the proof we fix $t = \frac{2k-1}{k-1}$.

Lemma 3.2.5. $R \geq L_l + L_{\bar{l}} + \frac{H+M}{t}$.

Proof. We divide the analysis into two cases. Suppose first that $L_{\bar{l}} \cdot t \geq H$, which implies that $\frac{k-1}{k} \cdot (t-1) \cdot L_{\bar{l}} = L_{\bar{l}} \geq \frac{H}{t}$ by our choice of t. We have that:

$$R_f \geq L_l + \frac{M}{t} + \frac{k-1}{k} \cdot t \cdot L_{\bar{l}} + \frac{1}{k} \cdot L_{\bar{l}} \geq L_l + \frac{M}{t} + L_{\bar{l}} + \frac{k-1}{k} \cdot (t-1) \cdot L_{\bar{l}} \geq L_l + L_{\bar{l}} + \frac{H+M}{t}$$

Suppose now that $L_{\bar{l}} \cdot t < H$:

$$R_p \geq L_l + \frac{M}{t} + \frac{k-1}{k} \cdot H + \frac{1}{k} \cdot L_{\bar{l}} \geq L_l + \frac{M}{t} + \frac{H}{t} + \frac{k^2-2k+1}{k(2k-1)} \cdot H + \frac{1}{k} \cdot L_{\bar{l}} > L_l + \frac{M}{t} + \frac{H}{t} + \frac{k-1}{k} \cdot L_{\bar{l}} + \frac{1}{k} \cdot L_{\bar{l}}$$

$$= L_l + L_{\bar{l}} + \frac{H+M}{t}$$

□

We are now finally able to analyze the ratio between the expected revenue of the revenue-maximizing auction and the k-lookahead auction. We consider two cases and show that in each one the expected ratio is at most $2 - \frac{1}{t} = \frac{3k-1}{2k-1}$. In the first case we assume that $L_l + L_{\bar{l}} \leq (H + M)(1 - \frac{1}{t})$. Therefore,

$$\frac{OPT}{R_h} \leq \frac{L_l + L_{\bar{l}} + H + M}{H + M} \leq \frac{(H + M)(1 - \frac{1}{t}) + H + M}{H + M} = 2 - \frac{1}{t}$$

Now assume that $L_l + L_{\bar{l}} > (H + M)(1 - \frac{1}{t})$. We have that, using Lemma 3.2.5:

$$\frac{OPT}{R} \leq \frac{L_l + L_{\bar{l}} + H + M}{L_l + L_{\bar{l}} + \frac{H+M}{t}} < \frac{(H + M)(1 - \frac{1}{t}) + H + M}{(H + M)(1 - \frac{1}{t}) + \frac{H+M}{t}} = 2 - \frac{1}{t}.$$

□

3.3 Reserve-price-based Auctions for Independent Distributions

The lookahead auction was developed as a mechanism to approximate optimal revenue under correlated distributions. However, in this section, we will see that it is powerful in analyzing *reserve-based* variants of the VCG auction. These auctions allocate the item(s) in a manner that maximizes the social welfare, except that the bidders' values have to be above a certain threshold called the *reserve price*[2]. These auctions are probably the most ubiquitous in practice, at least much more common than the optimal auction given by Myerson [43] (Theorem 2.0.1). Hartline and Roughgarden [32] pioneered the study of revenue guarantees of such auctions. It is reassuring that under fairly general assumptions such auctions are shown to generate near-optimal revenues.

As indicated below, most results on reserve-based auctions shown in this section are rederivations of previously known theorems from Hartline and Roughgarden [32], Dhangwatnotai et al. [21] and Azar et al. [4]. The main contribution of this section is

[2]When the same reserve price is used for all bidders, it can be interpreted as the value of the auctioneer for the item being auctioned, and the VCG auction with a reserve is genuinely a social welfare maximizing mechanism in this case.

a unified framework via lookahead auctions that considerably simplifies these results. Also, we develop a generalization of the lookahead auction to matroid settings in Section 3.3.3.

3.3.1 Preliminaries

We first define a pair of properties for valuation distributions that are strictly more general than regularity. Most results presented in this section will apply to distributions satisfying these properties.

Definition 3.3.1 (Prepeak Monotonicity)**.** A valuation distribution is said to be *prepeak monotone* if the revenue curve is an increasing function for q on $[0, q^*]$, where q^* is the quantile corresponding to the monopoly reserve.

In other words, a distribution is prepeak monotone if the expected revenue monotonically decreases as the price rises above the monopoly reserve. Prepeak monotonicity is a strictly weaker property than regularity. On one hand, prepeak monotonicity does not require any condition on the revenue curve to the right of its highest point; on the other hand, to the left of its peak, only monotonicity but not concavity is required.

Symmetrically, we say a distribution is *postpeak monotone* if the revenue curve is a decreasing function for q on $[q^*, 1]$. In other words, before the price reaches the monopoly reserve, the expected revenue monotonically increases as one raises the price. Postpeak monotonicity is also a strictly weaker condition than regularity.

There are two subtle variants of the VCG auctions depending on when the reserves are applied. Denote by r_i the *reserve price* for bidder i.

Definition 3.3.2 (VCG Auctions with Lazy Reserve Prices (VCG-L))**.** In the VCG auction with *lazy* reserve prices, the mechanism first chooses a tentative subset W of winners as in the VCG auction, i.e., $W = \mathrm{argmax}_{S\in\mathcal{I}} \sum_{i\in S} v_i$. Then each bidder $i \in W$ is presented a take-it-or-leave-it price set to be the higher of r_i and p_i^{VCG}.

We note that VCG-L is well-defined only for downward-closed settings, since we require any subset of W to be feasible.

Definition 3.3.3 (VCG Auctions with Eager Reserve Prices (VCG-E))**.** In the VCG auction with *eager* reserve prices, all bidders whose valuations are below their respective reserve prices are excluded from the auction. Then the VCG auction is run among the remaining bidders, with the payment for winner i set to be the higher of r_i and p_i^{VCG}.[3]

Both variants of the VCG auction are dominant strategy incentive compatible. (In particular, in VCG-E, a bidder i with $v_i < r_i$ has no incentive to pretend to have a higher value in order to avoid elimination at the start of the auction, because even if he does so and wins the auction, he will face a payment of at least r_i, which renders him a negative utility.) When the reserve price for each bidder is set to be the monopoly reserve of that bidder, i.e., $r_i = r_i^{\mathrm{mon}}, \forall i$, the corresponding auctions are called *VCG auctions with monopoly reserve prices*, denoted as VCG-LM and VCG-EM, respectively, for lazy and eager reserve prices.

It is easy to see that, in a single-item auction, if a bidder eventually wins (i.e., takes the service at the offered price) in VCG-L, he also wins in VCG-E, while the reverse is not true. In general, we have:

Proposition 3.3.1. *When using the same reserve prices, VCG-E results in weakly more social welfare than VCG-L in auctions with downward-closed feasible constraints.*

Proof. Recall from Chapter 2 that $\mathcal{I}$ is the set of feasible sets. Fix the valuation profile $v_1, \ldots, v_n$, and reserve prices $r_1, \ldots, r_n$, let N' be $\{i \in [n] \mid v_i \geq r_i\}$, and let $\mathcal{I}'$ be $\{T \subseteq [n] \mid T = S \cap N', S \in \mathcal{I}\}$. By downward closure, VCG-E simply allocates

[3]Both the winning set and the VCG payments here are calculated among the remaining bidders only.

to bidders in a set T that maximizes the sum of valuations over $\mathcal{I}'$, whereas VCG-L tentatively chooses a set S that maximizes the sum of valuations over $\mathcal{I}$, and then allocates to the bidders in $S \cap N'$, which is also a set in $\mathcal{I}'$. It is clear that the social welfare in VCG-E is at least that in VCG-L. □

3.3.2 Single Item Settings

In this section we first look at the implications of lookahead auctions for single-item auctions with independently drawn valuations. In Section 3.3.3 we will generalize all results in this section. In this section we will use the terms "second price auction" and "VCG auction" interchangeably.

Our first step is a comparative study of revenues from VCG-L and VCG-E. As we will see later, VCG-L in general is easier to analyze by way of the lookahead auctions, and the following theorem enables us to pass approximation guarantees for VCG-L to VCG-E as well.

Theorem 3.3.2. *In a single item auction where bidders' valuations are drawn independently from prepeak monotone distributions, the second price auction with eager monopoly reserve prices (VCG-EM) generates weakly more revenue than the second price auction with lazy monopoly reserve prices (VCG-LM).*

Proof. The key idea of the proof is to consider the expected revenue from *each* bidder, conditioning on the other bidders' valuations. We calculate the conditional expected revenues from the two auctions and compare them. The computation of the conditional revenue does not assume conditions on the reserve price or the valuation distribution; only in the last step do we use the property of monopoly reserves and the valuation distribution's prepeak monotonicity.

Fix a bidder i, we condition on all other bidders' valuations. Denote by h^{L}_{-i} the highest valuation among these other bidders, i.e., $h^{\mathrm{L}}_{-i} = \max_j\{v_j \mid j \neq i\}$. Denote by h^{E}_{-i}

the highest valuation among bidders except i who bid above their reserve prices, i.e., $h^{\mathrm{E}}_{-i} = \max_j\{v_j \mid j \neq i, v_j \geq r_j\}$. Clearly, $h^{\mathrm{L}}_{-i} \geq h^{\mathrm{E}}_{-i}$.

When $h^{\mathrm{L}}_{-i} \leq r_i$, the revenue extracted from bidder i is exactly the same in VCG-E and VCG-L, because in both auctions, bidder i will make a payment if and only if his bid is at least r_i, in which case he pays r_i. Therefore we need only consider the case $h^{\mathrm{L}}_{-i} > r_i$.

In VCG-L, bidder i makes a payment of h^{L}_{-i} if and only if his valuation is above h^{L}_{-i} (since $h^{\mathrm{L}}_{-i} > r_i$). The expected revenue from him is then $h^{\mathrm{L}}_{-i}(1 - F_i(h^{\mathrm{L}}_{-i}))$. In VCG-E, bidder i makes a payment if and only if his valuation is above both r_i and h^{E}_{-i}, and the payment is $p^{\mathrm{E}}_i = \max\{r_i, h^{\mathrm{E}}_{-i}\}$. The expected revenue from i is therefore $p^{\mathrm{E}}_i(1 - F_i(p^{\mathrm{E}}_i))$. Note that $h^{\mathrm{L}}_{-i} \geq \max\{r_i, h^{\mathrm{E}}_{-i}\} = p^{\mathrm{E}}_i$.

But the revenues $h^{\mathrm{L}}_{-i}(1 - F_i(h^{\mathrm{L}}_{-i}))$ and $p^{\mathrm{E}}_i(1 - F_i(p^{\mathrm{E}}_i))$ are simply the expected revenue we get by setting a price of h^{L}_{-i} or p^{E}_i to bidder i, respectively. By prepeak monotonicity of the valuation distribution, the expected revenue monotonically decreases as we raise the price above the monopoly reserve. Since $h^{\mathrm{L}}_{-i} \geq p^{\mathrm{E}}_i \geq r_i = r^{\mathrm{mon}}_i$, we have

$$h^{\mathrm{L}}_{-i}(1 - F_i(h^{\mathrm{L}}_{-i})) \leq p^{\mathrm{E}}_i(1 - F_i(p^{\mathrm{E}}_i)).$$

The above is a conditional analysis, but we see that the expected revenue in VCG-LM from each bidder i is no more than the expected revenue in VCG-EM from the same bidder, no matter what the other bidders bid. Our theorem immediately follows. □

We now develop a connection between VCG-L and the lookahead auction, and explore its consequences.

Theorem 3.3.3. *The lookahead auction extracts at least as much revenue as VCG-L with any reserve prices. When bidders' valuations are drawn independently from prepeak monotone distributions, the lookahead auction and the second price auction with lazy monopoly reserve prices are identical.*

Proof. Since VCG-L only sells to and possibly charges the highest bidder, from whom the lookahead extracts the optimal revenue, the first part of the theorem is obvious. We now look at the second part.

By definition, the lookahead auction runs the optimal auction for the highest bider, conditioning on that his valuation is higher than all other bidders'. [4] Since the optimal auction for any single bidder is simply to set a take-it-or-leave-it price [43], we only need to see what price we should set for the highest bidder.

Let i be the highest bidder, and denote by h_{-i} the second highest bid. The distribution of v_i, conditioning on $v_i \geq h_{-i}$ is just F_i truncated at h_{-i}, i.e., $F_i^{\text{cond}}(v) = \frac{F_i(v)-F(h_{-i})}{1-F_i(h_{-i})}$, for any $v \geq h_{-i}$, and $F_i^{\text{cond}}(v) = 0$ for any $v < h_{-i}$. Setting any price below h_{-i} would be obviously suboptimal, and the revenue of setting a price of $p \geq h_{-i}$ is

$$p(1 - F_i^{\text{cond}}(p)) = p \cdot \left(1 - \frac{F_i(p) - F_i(h_{-i})}{1 - F_i(h_{-i})}\right) = p(1 - F_i(p)) \cdot \frac{1}{1 - F_i(h_{-i})}.$$

In words, for prices above h_{-i}, the conditional expected revenue is simply scaled up by a constant factor of $\frac{1}{1-F_i(h_{-i})}$ from the unconditioned distribution. Therefore, if $h_{-i} < r_i^{\text{mon}}$, the optimal price to set for the conditional distribution is still r_i^{mon}; and if $h_{-i} \geq r_i^{\text{mon}}$, by prepeak monotonicity, the expected revenue monotonically decreases as the price rises above h_{-i}; therefore the optimal price to set is h_{-i} itself.

To summarize, in the lookahead auction, we temporarily pick the highest bidder, and set a take-it-or-leave-it price that is the higher of r_i^{mon}, the monopoly reserve price, and h_{-i}, the second highest bid. This is exactly the second price auction with lazy monopoly reserve prices. □

Theorem 3.3.3 combined with Theorem 3.1.1 and Theorem 3.3.2 immediately gives the following corollary, a special case of Theorem 3.16 in Dhangwatnotai et al. [21] and Theorem 3.7 in Hartline and Roughgarden [32].

[4] Unlike the correlated settings considered by Ronen [48], here we assume independence, and so conditioning on the other bidders' valuations has no effect on the distribution of the highest bidder.

Corollary 3.3.4 (21, 32)**.** *For a single item auction, the second price auction with either lazy or eager monopoly reserve prices extracts at least half of the optimal revenue for bidders whose valuations are drawn independently from prepeak monotone distributions.*

We will see in Section 3.3.3 that our approach easily generalizes to matroid settings, matching the original theorems. The proof presented here, especially that for VCG-L (for which we do not need Theorem 3.3.2), is considerably shorter and less involved than the original proofs in the literature. In particular, we do not appeal to Myerson [43]'s characterization of optimal auctions with virtual valuations.

Remark 3.3.5. The theorems in Dhangwatnotai et al. [21] and Hartline and Roughgarden [32] are stated and proved for bidders with regular distributions, although their proof easily generalizes to prepeak monotone distributions.

3.3.3 Matroid Settings

In this section we extend all results from Section 3.3.2 to the more general matroid settings. We first generalize Theorem 3.3.2 to matroid settings.

Theorem 3.3.6. *In a matroid setting where bidders' valuations are drawn independently from prepeak monotone distributions, VCG auctions with eager monopoly reserves extract weakly more revenue than VCG auctions with lazy monopoly reserves.*

Proof. As in the proof of Theorem 3.3.2, we focus on the expected revenue from a fixed bidder i, conditioning on all other bidders' valuations. Let p_i^{VCGL} and p_i^{VCGE} denote the VCG payment for bidder i in VCG-L and VCG-E, respectively. (Note that the actual payment to be made by bidder i in these auctions is the higher of the reserve r_i and the VCG threshold; but we will focus mostly on the VCG thresholds in this proof.) The key step is to show $p_i^{\mathrm{VCGL}} \geq p_i^{\mathrm{VCGE}}$. Recall that in VCG-E, bidders with valuations below their reserve prices are excluded from both the allocation and payment calculation. Effectively, such bidders' bids are replaced by 0 in the calculation of VCG payments. This constitutes the only difference in computing p_i^{VCGL} and p_i^{VCGE}. It therefore suffices to show that bidder i's VCG payment weakly decreases as other bidders' bids are zeroed out.

We show this by expressing p_i^{VCGL} and p_i^{VCGE} in terms of a submodular function. Define a set function $f : 2^{[n]} \to \mathbb{R}$ by

$$f(S) = \max_{T \subseteq S, T \in \mathcal{I}} \sum_{j \in T} v'_j,$$

where $v'_j = v_j$ for $j \neq i$ and $v'_i = \sum_{j \neq i} v_j + 1$. Then $p_i^{\mathrm{VCGL}} = f([n] \setminus \{i\}) - (f([n] - v'_i)$. This is because $f([n] \setminus \{i\})$ represents the social welfare of the other bidders if i were not present, and $f([n]) - v'_i$ represents the social welfare of the other bidders if i were to be a winner (v'_i is set high enough such that the optimal set has to include i). The difference between the two is then the externality that i imposes on the other bidders by his winning, and therefore is equal to the VCG threshold p_i^{VCGL}. Similarly, let U denote the set of bidders whose valuations are above their reserve prices, i.e., $U = \{j \mid v_j \geq r_j\}$, then $p_i^{\mathrm{VCGE}} = f(U) - (f(U \cup \{i\}) - v'_i)$.

f is defined to be the result of a linear maximization over a matroid, and is well known to be submodular. Therefore, as $U \subseteq [n]$, $f([n]) - f([n] \setminus \{i\}) \leq f(U \cup \{i\}) - f(U)$. This gives $p_i^{\mathrm{VCGL}} \geq p_i^{\mathrm{VCGE}}$.

With this, the proof to Theorem 3.3.2 easily generalizes by replacing h^{L}_{-i} and h^{E}_{-i} with p_i^{VCGL} and p_i^{VCGE}, respectively, and we omit the rest of the proof. □

Remark 3.3.7. For more general environments (even downward closed), it need not be true that p_i^{VCGL} is at least p_i^{VCGE}. For example, consider $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}\}$. In this feasibility system, bidder 1's VCG payment weakly increases when v_2 decreases. So p_1^{VCGL} can be lower than p_1^{VCGE}.

In order to generalize Theorem 3.3.3, we first define a generalization of lookahead auctions for downward-closed settings.

Definition 3.3.4. The *lookahead auction for a downward closed setting* first selects a tentative set W of winners as in the VCG auction, i.e., $W = \text{argmax}_{S \in \mathcal{I}} \sum_{i \in S} v_i$, and then runs an optimal auction for each bidder i in W conditioning on (a) all other bidders' valuations and (b) the fact that bidder i's valuation is above the VCG threshold p_i^{VCG}.

We first generalize Theorem 3.1.1 to matroid settings.

Theorem 3.3.8. *The lookahead auction for matroid settings is a truthful mechanism that obtains at least half of the optimal revenue, even when the bidders' valuations are correlated.*

To prove Theorem 3.3.8, we will need the next well-known theorem on matroids. (See, e.g., [50], for a proof.)

Theorem 3.3.9. *Let B_1 and B_2 be any two independent sets of a matroid $\mathcal{M}$ such that $|B_1| = |B_2|$. There exists a bijective mapping $g : B_1 \setminus B_2 \to B_2 \setminus B_1$ such that $\forall e \in B_1 \setminus B_2$, $B_1 \setminus \{e\} \cup \{g(e)\}$ is independent in $\mathcal{M}$.*

Proof of Theorem 3.3.8. First, the lookahead auction is incentive compatible for similar reasons as in the single-item auction: a bidder not in W has no incentive to raise his bid to enter W, because the price he faces in the second stage will be at least the valuation threshold to enter W.

We then show the revenue guarantee. Denote by Rev_{LA} the expected revenue of the lookahead auction. The revenue of any optimal auction can be split into two parts, H and L: H is the expected revenue from W, and L the revenue from the rest of the bidders. Note that W here is random, determined by the realization of bidders' valuations, but H and L are expected values and not random. It suffices to show that Rev_{LA} is no less than both H and L.

Rev_{LA} is clearly at least H, since the lookahead auction runs the optimal auction for bidders in W, using all information available at that stage.

Let $T \subseteq [n] \setminus W$ be an independent subset that maximizes the social welfare among bidders not in W. The expectation of $\sum_{j \in T} v_j$ is an upper bound for L, since the auction cannot charge more than the winning bidders' valuations. Therefore it suffices to show $\text{Rev}_{\text{LA}} \geq \mathbf{E}[\sum_{j \in T} v_j]$. Since $|T| \leq |W|$, we can find a subset $U \subseteq W$ such that $U \cup T$ is independent and $|U \cup T| = |W|$. By Theorem 3.3.9 there exists a bijective mapping $g : W \setminus U \to T$ such that for any bidder i in $W \setminus U$, $W \setminus \{i\} \cup \{g(i)\}$ is independent. Therefore, the VCG threshold for each $i \in W \setminus U$ is at least $v_{g(i)}$. In the second stage of the lookahead auction, the auctioneer could simply set the VCG payment for each bidder in W and secure a revenue of at least $\sum_{j \in T} v_j$. By the optimality of the revenue from W in the lookahead auction, we have $\text{Rev}_{\text{LA}} \geq \mathbf{E}[\sum_{j \in T} v_j] \geq L$. □

Theorem 3.3.3 extends straightforwardly to general downward-closed settings since all arguments therein concern a single bidder. We omit its proof:

Theorem 3.3.10. *In any downward closed setting, the lookahead auction gets at least as much revenue as VCG-L with any reserve prices. Moreover, when bidders' valuations are drawn independently from prepeak monotone distributions, the lookahead auction and the VCG auction with lazy monopoly reserve prices are identical.*

Corollary 3.3.4 now generalizes to matroid settings, which amounts to a proof of both Theorem 3.16 in Dhangwatnotai et al. [21] and Theorem 3.7 in Hartline and Roughgarden [32].

Corollary 3.3.11 (21, 32)**.** *In matroid settings where bidders' valuations are drawn independently from prepeak monotone distributions, the VCG auction with lazy monopoly reserve prices gives at least half of the optimal revenue.*

Since the lookahead auctions have a very simple guiding philosophy, the conceptual connection between VCG-L and lookahead auctions promises to greatly simplify the analysis of VCG-L in general. As another application of this connection, we give a considerably shorter and elementary proof for Theorem 3.1 in Azar et al. [4], which is a major building block in that work.

Let $R_i(p)$ denote the expected revenue from bidder i by setting a price of p.

Theorem 3.3.12 (Theorem 3.1 in 4)**.** *For each bidder i, let r_i be a price drawn randomly from a certain distribution and independently from the valuation v_i, such that in the single-bidder setting, $\mathbf{E}_{r_i}[R_i(r_i)] \geq \alpha R_i(r_i^{\text{mon}})$. Then in all downward closed settings where bidders' valuations are drawn indepdendently from postpeak monotone distributions, VCG-L with random reserve prices $(r_1, \ldots, r_n)$ gets at least α-fraction of the revenue of VCG-L with monopoly reserve prices (VCG-LM).* [5]

Proof. Fix a tentative winner $i \in W$, and condition on the VCG price p_i^{VCG}. For a reserve price r_i, let ρ denote the ratio $R_i(r_i)/R_i(r_i^{\text{mon}})$, then we have $\mathbf{E}[\rho] \geq \alpha$. Let ρ^{cond} denote the ratio between the expected VCG-L revenue from bidder i using reserve price r_i and that from VCG-LM, conditioning on $v_i \geq p_i^{\text{VCG}}$. It suffices to show $\mathbf{E}[\rho^{\text{cond}}] \geq \alpha$. We will show $\rho^{\text{cond}} \geq \rho$ pointwise. [6] Since, as we have shown in the proof of Theorem 3.3.3, the revenue from all prices above p_i^{VCG} is scaled up by a factor of $1/(1 - F_i(p_i^{\text{VCG}}))$ when conditioning on $v_i \geq p_i^{\text{VCG}}$, for $r_i \geq p_i^{\text{VCG}}$ we have

$$\rho^{\text{cond}} = \frac{R_i(r_i)/(1 - F_i(p_i^{\text{VCG}}))}{R_i(\max\{p_i^{\text{VCG}}, r_i^{\text{mon}}\})/(1 - F_i(p_i^{\text{VCG}}))} \geq \frac{R_i(r_i)}{R_i(r_i^{\text{mon}})} = \rho.$$

For $r_i < p_i^{\text{VCG}}$, if $p_i^{\text{VCG}} \geq r_i^{\text{mon}}$, then both auctions will use p_i^{VCG}, and $\rho^{\text{cond}} = 1 \geq \rho$; if $p_i^{\text{VCG}} < r_i^{\text{mon}}$, we have

$$\rho^{\text{cond}} = \frac{R_i(p_i^{\text{VCG}})/(1 - F_i(p_i^{\text{VCG}}))}{R_i(r_i^{\text{mon}})/(1 - F_i(p_i^{\text{VCG}}))} = \frac{R_i(p_i^{\text{VCG}})}{R_i(r_i^{\text{mon}})} \geq \frac{R_i(r_i)}{R_i(r_i^{\text{mon}})} = \rho.$$

[5] Azar et al. [4] stated the theorem for regular distributions, though their proof generalizes to postpeak monotone distributions.

[6] To clarify, ρ is a random variable that depends only on the value of r_i, whereas ρ^{cond} is a random variable that depends on both r_i and p_i^{VCG}. To prove $\rho^{\text{cond}} \geq \rho$ "pointwise", we are going to show that, for any value of r_i and p_i^{VCG}, $\rho^{\text{cond}} \geq \rho$.

The last inequality comes from postpeak monotonicity, i.e., $r_i < p_i^{\mathrm{VCG}} \leq r_i^{\mathrm{mon}}$ implies $R_i(r_i) \leq R_i(p_i^{\mathrm{VCG}})$. This completes the proof. $\square$

CHAPTER 4

MARGINAL REVENUE MECHANISMS

4.1 Introduction

Marginal revenue plays a fundamental role in microeconomic theory. For example, a monopolist providing a commodity to two markets each with its own concave revenue (as a function of the supply provided to that market) optimizes her profit by dividing her total supply to equate the marginal revenues across the two markets. Moreover this central economic principle also governs classical auction theory. Myerson [43] characterizes profit maximizing single-item auction as formulaically optimizing the *virtual value* of the winner; Bulow and Roberts [9] reinterpret Myerson's virtual value as the marginal revenue of a certain concave revenue curve.

Because it is simple and intuitive, the Myerson-Bulow-Roberts approach provides the basis for much of Bayesian auction theory. Unfortunately though, the theory is limited to settings where agents have linear single-dimensional preferences, i.e., where an agent's utility is given by her value for service less her payment. Consequently, Bayesian auction theory is often similarly limited. With more general forms of agent preferences, especially multi-dimensionality, e.g., for multi-item auctions, or non-linearity, e.g., risk aversion or budgets, auction theory is complex, less versatile, and often not well understood.

Our main result in this chapter is to show that hidden under the complexity of optimal mechanism design problems for agents with multi-dimensional and non-linear (henceforth: general) preferences is marginal revenue maximization. The approach of marginal revenue maximization expresses a multi-agent mechanism design problem as a composition of simple single-agent mechanism design problems, i.e., from the construction of the appropriate notion of revenue curves. This new approach for general preferences uncovers a condition we refer to as *revenue linearity* that contains all linear single-dimensional preferences and governs the performance of the marginal revenue mechanism. When the single agent problems are revenue linear, marginal revenue maximization is optimal and the Myerson-Bulow-Roberts mechanism generalizes exactly. When the single agent problems are approximately revenue linear, marginal revenue maximization is approximately optimal (though the composition of the single agent mechanisms to implement marginal revenue maximization requires new techniques). Finally, because our marginal revenue approach is structurally similar to the classical approach, many results from classical auction theory approximately and automatically extend to general preferences.

A central result to classical auction theory comes from reinterpreting the Myerson-Bulow-Roberts mechanism (i.e., for maximizing marginal revenue) in the special case of symmetric agents. As an example of the benefits of our approach, compare this classical reinterpretation with a similar reinterpretation of our results. In the classical setting there is a single item for sale and agents with i.i.d. values for it; in our setting there is a single item for sale which the seller can configure on one of several ways and agents have i.i.d. values for each configuration, e.g., a car that can be painted red or blue (importantly, the seller sets the configuration and the buyer cannot change it).

Selling a car Classical auction theory says that (a) the optimal way to sell an object (henceforth: a car) to a single agent with value drawn from a uniform distribution on $[0, 1]$ is to post a take-it-or-leave-it price of 1/2, (b) the optimal way to sell a car to one of multiple agents with uniformly distributed values is to run a second-price auction with reserve price 1/2, and (c) more generally the optimal way to sell the car to multiple agents with i.i.d. values is to run the second price auction with the same reserve price that would be offered as a take-it-or-leave-it price to one agent (assuming the distribution satisfies some mild assumptions).

Selling a red-or-blue car Consider selling a car that on sale can be painted one of two colors, red or blue.[1] Our theory says that (a) the optimal way to sell a red-or-blue car to a single agent with values for the different colors each drawn independently and uniformly from $[0, 1]$ is to post a take-it-or-leave-it price of $\sqrt{1/3}$ for either color, (b) the optimal way to sell a red-or-blue car to one of multiple agents each with i.i.d. uniform values for each color is to run the second-price auction with reserve $\sqrt{1/3}$ and allow the winning agent to choose her favorite color on sale, and (c) more generally to sell a red-or-blue car to one of multiple agents each with values drawn i.i.d. (from a distribution that satisfies the same mild assumptions as above) for each color, the second price auction with the reserve price equal to the same price that would be offered to a single agent is (at worst) a 4 approximation to the optimal auction.

It should be noted that reducing the multi-dimensional preference to a single-dimensional preference by always selling the winning agent her favorite color is very natural and practical; however, it is not generally optimal beyond $U[0, 1]$. Even for a single agent with values for both colors distribution uniformly on $[5, 6]$, an analysis of Thanassoulis [52] shows that the optimal auction does not sell the agent her favorite item subject to a reserve (in fact, it is not even deterministic).

Approach We focus on *service constrained environments* where in any outcome the mechanism produces, each agent is either considered served or unserved. The designer has a feasibility constraint that governs which subset of agents can be simultaneously served, but other aspects of the outcome, e.g., payments, are unconstrained. This model allows additional unconstrained attributes of the service (e.g., the color of the car in the previous red-or-blue car example). We assume that the space of mechanisms is closed

[1] While the reserve prices given are for two colors, with the appropriate reserve price the approximation bounds hold for any number of colors.

under convex combination which allows for randomized mechanisms.

The agents in the mechanism have independently but not necessarily identically distributed preferences (a.k.a., types). We do not place any assumption on the agent preferences other than they are expected utility maximizers. This includes the most challenging preference models in Bayesian mechanism design such as multi-dimensionality, public or private budgets, and risk-aversion (e.g., as given by a concave utility function).

Revenue curves defined in Chapter 2 can be generalized by the following single-agent mechanism design problem. Consider a single agent with preferences drawn from a known distribution. Via the taxation principle (e.g., 55) the outcomes of a mechanism, for all possible preference reports the agent might make, can be viewed as a menu where the agent selects her favorite outcome by making the appropriate report. Of course with different preferences the agent would potentially select different outcomes. This menu may contain outcomes that are randomized and for this reason we refer to it as a *lottery pricing*. Ex ante, i.e., in expectation over the distribution of the agent's preference, a lottery pricing induces a probability with which the agent receives an outcome that corresponds to service, and an expected payment, i.e., revenue.

As every lottery pricing induces an ex ante service probability and revenue, we can ask the optimization question of identifying the lottery pricing with a given ex ante service probability that has the highest expected revenue. We can also consider this optimal revenue as a function of the ex ante service probability, giving rise to the agent's *revenue curve*. Important in the construction of revenue curves are the lottery pricings, i.e., single-agent mechanisms, that give the optimal revenue for each ex ante service probability. As the space of (mechanisms and hence) lottery pricings is closed under convex combination, the revenue curves are always concave. The marginal revenue curve is the derivative of the revenue curve with respect to ex ante service probability.

As discussed in the opening paragraph, the standard economic intuition suggests that a monopolist splitting the sale of a commodity between two markets should do so to equate marginal revenue. There is an intuitive algorithmic reinterpretation of this fact. If we break the allocation to each market into tiny pieces and attribute to each piece the change in revenue from adding that piece (i.e., the marginal revenue), then the total revenue of an allocation is the sum of the marginal revenues of each piece. A simple algorithm for optimizing this cumulated marginal revenue is to repeatedly allocate a tiny amount to the market that has the highest marginal revenue at its current allocation (until the good is totally allocated or marginal revenues are non-positive). Clearly this results in a final allocation where the markets marginal revenues are roughly equal as in the microeconomic interpretation. This allocation is optimal.

Our main contribution is a methodology for constructing multi-agent mechanisms from the simple single-agent lottery pricings that define the revenue curve. The main task of such a construction is to specify a method for combining the single agent mechanisms into a multi-agent mechanism that is both feasible with respect to the service constraint and obtains good revenue. We refer to the family of mechanisms that take the following form as *marginal revenue mechanisms*.

1. Map each agent type (which may lie in an arbitrary type space) to a *quantile* in $[0, 1]$.
2. Calculate the marginal revenue of each agent as the derivative of the revenue curve at her quantile.
3. Select for service the set of agents that maximize cumulative marginal revenue subject to feasibility.
4. Calculate for each agent the appropriate non-service aspects of the outcome, e.g., payments.

Thus far in the discussion only steps 2 and 3 should be clear. The remaining steps are non-trivial in general and a main issue that we will be resolving.

Results This chapter generalizes the marginal-revenue approach for agents with single-dimensional linear preferences which is due to Bulow and Roberts [9] to general preferences. Our main algorithmic contribution is to generalize Steps 1 and 4 thereby reducing service constrained multi-agent mechanism design problems to (single agent) ex ante constrained lottery pricing problems. There are a number of challenges in this endeavor. First, revenue equivalance does not hold for general preferences (which is used in the proof of optimality for single-dimensional preferences). Second, there is not a natural ordering on preferences for general preferences (making it difficult to map preferences to quantiles). Third, the set of agents served by the marginal revenue mechanism maybe randomized. None of these issues are present for single-dimensional linear preferences. Finally, the reduction focuses attention on this lottery pricing problem as a fundamental building block of good mechainsms. For general preferences these lottery pricing problems have not previously been considered in the literature.

Orthogonal to the question of implementing the marginal revenue mechanism for general preferences are questions of quantifying its performance. Via the Myerson-Bulow-Roberts analysis it is known that for single-dimensional linear preferences, the marginal revenue mechanism is optimal. As a first step in understanding the performance of the mechanism more generally we give a new rederivation of the optimality for single-dimensinal agents that exposes a previously unobserved property of single-dimensional preferences which we refer to as *revenue linearity*. The optimality of the marginal revenue mechanism is implied by revenue linearity. Moreover, if the single-agent problems are approximately revenue linear (e.g., bounded from below by a linear function and from above by α times the function), then marginal revenue maximization is an α-approximation to the optimal mechanism.

Revisiting our red-or-blue car examples above, (a) is a description of the optimal unconstrained lottery pricing, (b) is a consequence of the revenue-linearity of types that are uniformly distributed on a multi-dimensional hypercube, and (c) is a consequence of 4-approximate revenue linearity for agents with types drawn from any product distribution.

One of the main benefits of considering the marginal revenue mechanism for approximately optimal mechanism design is that, as its structure is similar to optimal mechanisms for single-dimensional environments, many results from the extensive literature can be easily generalized. The following are some of the most important consequences.

algorithmic mechanism design When weighted optimization is hard we can replace an exact algorithm for weighted maximization with any approximation algorithm using either of the single-dimensional black-box reductions of Hartline and Lucier [31] and Hartline et al. [33].

sequential posted pricing Sequential posted pricing mechanisms of Chawla et al. [16] and Yan [56] that are approximately optimal for single-dimensional agents are approximately optimal for general agents (in the same service constrained environment) and the same approximation factor is guaranteed. Moreover, these sequential posted pricing bounds give another bound on the approximation factor of the marginal revenue mechanism. The marginal revenue mechanism is in fact optimal within a class of mechanisms that contains the sequential posted pricing mechanisms; therefore, its approximation factor is no worse. As an example, for the single-item service constraint, a sequential posted pricing bound implies an $e/(e-1)$-approximation regardless of approximate linearity of the lottery pricing problems.

simple versus optimal While our marginal revenue mechanism is already generally much simpler than the optimal mechanism, we can get even simpler approximation mechanisms by appling methods for proving that simple mechanisms approximate the marginal revenue mechanism that have been developed for single-dimensional preferences. In particular, Hartline and Roughgarden [32] show that in single-dimensional environments maximizing marginal revenue is more complex than simple reserve-price-based mechanisms, i.e., mechanisms that maximize welfare subject to a reserve price. Nonetheless, they show that reserve-price-based mechanisms are often approximately optimal. When the structure of the optimal lottery pricings is simple, e.g., in generalizations of the red-or-blue car example, these mechanisms extend to general preferences.

single-sample mechanisms Approaches above have been for Bayesian optimal mechanism design where the designer optimizes a mechanism given a distribution of preferences. Dhangwatnotai et al. [21] relax the assumption that the distribution is known and show that a mechanism based on drawing a single sample from the distribution gives a good approximation to the Bayesian optimal mechanism. Again, the single-sample framework extends to general preferences for which the structure of optimal lottery pricings is simple.

It is important to contrast the simplicity of the marginal revenue approach with recent algorithmic results in Bayesian mechanism design for general agent preferences. Recently, Alaei et al. [1] and Cai et al. [11, 12, 13] gave polynomial time mechanisms for large important classes of Bayesian mechanism deign problems; the former considered general preferences in service constrained settings (as does this chapter) and the latter considered multi-dimensional additive preferences. The two main conclusions of these works is that (a) optimal mechanisms continue to have weighted maximization at their core, and (b) the appropriate weights (i.e., virtual values) are stochastic and can be solved for as a convex optimization problem, e.g., via ellipsoid method, that takes into account the feasibility constraint and the distribution over types of all agents. (This latter result is simply because the space of mechanisms is convex, any point on the interior of a convex set can be implemented by a convex combination of vertices, and vertices correspond to linear, a.k.a., weighted, optimization.) There are a number of important distinctions between the results in this chapter and these algorithmic results. First, the weights in our derivation have a natural economic interpretation as marginal revenue. Second, the weights in our derivation can be found easily from solutions to the single-agent lottery pricing problems and are not derived from the solution to an additional optimization problem. Third, in most cases, the weights in our derivation depend only on the single-agent problem and not on the multi-agent feasibility constraint or presence of other agents. Therefore, our approach affords significant structural simplification and interpretation that enables the consequences previously enumerated. Finally, one of the biggest open questions in the above algorithmic work is in developing approaches that are not brute-force in each agent's type space. For example, our approach gives mechanisms for multi-dimensional unit-demand agents with values from a product distribution that are polynomial in the demensionality of the of the type space (logarithmic in the size of the typespace).

Organization of the Chapter In Section 4.2 we review the Myerson-Bulow-Roberts single-dimensional linear agent model, their approach to Bayesian optimal mechanism design, and give a new proof that the marginal revenue mechanism is revenue optimal. The proof follows from an argument that for single-dimensional linear agents a class of single-agent lottery pricing problems satisfies a natural revenue linearity property.

In Section 4.3 we formalize our service constrained model for general preferences and generalize the marginal revenue derivation to general preferences that satisfy the previously identified revenue linearity property. In Section 4.4 we extend the marginal revenue mechanism to general preferences regardless of revenue linearity. We show that approximate linearity implies approximate optimality and give general methods for implementing the marginal revenue mechanism (e.g., Steps 1 and 4).

4.2 Warmup: Single-dimensional Linear Preference

In this section we warm up by giving a new proof that the marginal revenue mechanism is revenue optimal for agents with single-dimensional linear preferences. In this proof we will introduce many concepts that make our generalization possible (which were not present in previous proofs), though in the simpler single-dimensional setting where they are more straightforward. The basic approach is as follows. We formulate an important class of lottery pricing problems the solution to which define a revenue curve. We show that single-dimensional agents are *revenue linear* in the sense that it is optimal to decompose the allocation to any agent as a convex combination of the solutions to these lottery pricing problems. Finally, we observe that this implies that the optimal revenue can be expressed in terms of the cumulative (over agents served) marginal revenue (given by the derivative of the revenue curve). The marginal revenue mechanism optimizes this latter term pointwise and, therefore, also in expectation. In the interest of brevity we will keep the discussion informal, the proof here subsumed by the generalization in Section 4.3 which we give formally.

Model We first give a more detailed and intuitive interpretation of the revenue curve defined in Chapter 2. The geometry of single-dimensional auction theory is more readily apparent when we index an agent's strength relative to the distribution (instead of values). Let $V(q) = F^{-1}(1-q)$ be the *inverse demand curve*, i.e., $V(\hat{q})$ is the posted price that would be accepted by the $\hat{q}$ measure of highest-valued agents (and rejected by all others). The *quantile* of an agent is the measure of agents with higher values, i.e., for value v the agent's quantile is $q = V^{-1}(v)$. Importantly, for v drawn at random from the distribution, $q = V^{-1}(v)$ is uniform on $[0, 1]$ (therefore, expectations of functions of q are given by integrals with probability density one).

A multi-agent mechanism design problem is given by n such single-dimensional agents each with their respective inverse demand curves (which may be distinct) and a feasibility constraint governing the subsets of agents that can be simultaneously served. In the interim stage, i.e., when an agent knows her own value but not the values of other agents, the mechanism looks to the agent like a single-agent mechanism. It will thus be sufficient for most of the analysis of optimal multi-agent mechanisms to consider the appropriate single-agent problems.

From the perspective of an agent in a single-agent mechanism and as a function of the agent's report, the agent is served with some probability and makes some expected payment. We can view this function as a menu of service probabilities and expected payments where the agent selects her favorite outcome by submitting the corresponding report. Notice that depending on the agent's value for service, she may choose different outcomes. We may as well index the outcomes in the menu by the quantile of the agent that selects the outcome, i.e., agent with quantile q chooses outcome $(x(q), q(q))$. We assume that outcome $(x, p) = (0, 0)$ is in the menu. This relabeling and assumption imply incentive compatibility and individual rationality, respectively, i.e.,

$$V(q)x(q) - q(q) \geq V(q)x(q') - q(q'), \qquad \forall q, q' \in [0, 1]. \tag{IC}$$

$$V(q)x(q) - p(q) \geq 0, \qquad \forall q \in [0, 1]. \tag{IR}$$

We call such a menu a *lottery pricing*. When the lottery pricing is induced in the interim stage of a multi-agent mechanism, then the constraints above are Bayesian incentive compatible and interim individual rational, as defined in Chapter 2.

The Myerson [43] characterization of Bayesian incentive compatible mechanisms applies to lottery pricings and implies that the *allocation rule* $x(\cdot)$ is monotone non-decreasing and the *payment rule* is given precisely as a function of $x(\cdot)$. An important consequence of the latter part of this characterization is *revenue equivalence*. We will make strong usage of both monotonicity and revenue equivalence below, though the specific form of the payment rule will not be important.

Constrained Lottery Pricings Given such lottery pricing and a distribution over the agent's value, an ex ante expected payment $\mathbf{E}_q[q(q)]$ and ex ante probability of service $\mathbf{E}_q[x(q)]$ are induced. The single-agent lottery pricing problem that forms the basis for the marginal revenue mechanism is the following. Given an ex ante constraint $\hat{q}$ on the probability with which the agent is served, find the lottery pricing that serves the agent with probability $\hat{q}$ and maximizes revenue.

Definition 4.2.1. The *revenue curve* $R(\hat{q})$ is defined for all $\hat{q} \in [0, 1]$ as the optimal lottery pricing revenue for ex ante constraint $\hat{q}$.

In order to show that convex combinations of optimal ex ante constrained lottery pricings are optimal in general, we need to consider a more generalized lottery pricing problem. Notice that the ex ante constraint lottery problem gives an (equality) constraint on the total probability that the agent is served over all quantiles she may have. To get more fine-grained control over the lottery pricing we additionally allow upper bounds to be specified on the total probability of allocation of a subset of quantiles. Consider the following lottery pricing problem: Given a monotone concave function $\hat{X}(q)$, find the optimal lottery pricing where the ex ante probability of allocating to any $\hat{q}$ measure of quantiles is at most $\hat{X}(\hat{q})$ for all $\hat{q} \in [0, 1)$ and exactly equal to $\hat{X}(\hat{q})$ at $\hat{q} = 1$.

To see why this constrained lottery pricing problem is the right one to consider, notice the following. First, because any allocation rule is monotone, meaning stronger quantiles receive no lower probability of service than weaker quantiles, the sets of measure $\hat{q}$ for which the constraint of service probability at most $X(\hat{q})$ is binding correspond exactly to the strongest $\hat{q}$ measure of quantiles. For allocation rule $x(\cdot)$ the probability of service to the strongest $\hat{q}$ measure of agents is exactly $X(\hat{q}) = \int_0^{\hat{q}} x(q)\,\mathrm{d}q$. We refer to $X(\cdot)$ as the *cumulative allocation rule*. Thus, the allocation constraint is exactly, $X(\hat{q}) \leq \hat{X}(\hat{q})$ for all $\hat{q} \in [0, 1]$ (with equality for $\hat{q} = 1$).

Of course we can view the cumulative allocation rule X of x as a constraint and observe that x satisfies the constraint with equality. Moreover, x is the allocation rule that satisfies X as a constraint that has the highest probability on stronger (i.e., lower) quantiles). Therefore, for any constraint $\hat{X}$ (with corresponding $\hat{x}(q) = \frac{\mathrm{d}}{\mathrm{d}q}\hat{X}(q)$) is met by allocation rule x that relatively has allocation probability shifted from stronger quantiles to weaker quantiles. Specifically, $\hat{x}$ *majorizes* x.

Definition 4.2.2. $\mathrm{Rev}[\hat{x}]$ is the optimal revenue of any lottery pricing that satisfies the allocation constraint $\hat{x}$ (via its cumulative allocation rule $\hat{X}$).

Recall our ex ante constrained lottery pricing where we wish to serve the agent with ex ante probability $\hat{q}$. A *posted price* is parameterized by a single price and is a simple example of a lottery pricing (i.e., one that is deterministic), the two menu items are to be served and pay the price or not to be served and pay nothing. The agent prefers service when her value exceeds the price and, otherwise, she prefers no service. For an agent with inverse demand curve $V(\cdot)$, the posted price that serves with probability $\hat{q}$ is $V(\hat{q})$. It gives expected revenue $\hat{q} \cdot V(\hat{q})$ (which is at most $R(\hat{q})$). Its allocation rule $\hat{x}^{\hat{q}}$ is the reverse step function that is one on quantiles $[0, \hat{q}]$ and then zero on $(\hat{q}, 1]$. This rule has the most service probability on strong quantiles of all allocation rules that satisfy the ex ante allocation constraint. Of course, the revenue it generates $\hat{q} \cdot V(\hat{q})$ may not be a concave function of $\hat{q}$ and it must be that the revenue curve $R(\cdot)$ is concave. It can be shown, in fact, that $R(\cdot)$ is exactly the concave hull of $\hat{q} \cdot V(\hat{q})$ and the optimal lottery for any $\hat{q}$ is given by a posted pricing or if $R(\cdot)$ is linear at $\hat{q}$ equal to the convex combination of two posted pricings (corresponding to the boundary of the interval containing $\hat{q}$ on which $R(\cdot)$ is linear). The allocation rule of this convex combination is a convex combination of the appropriate two reverse step functions and, in the sense described above, has service probability shifted from stronger quantiles to weaker quantiles. This specific form (which is not obvious) is not important for our rederivation of the optimal mechanism; what is important is that the optimal lotteries have weaker allocation rules than posted prices and that the revenue curve is at least the revenue of posted prices (both of which are obvious).

Revenue Linearity We are now ready to give the new derivation of the marginal revenue mechanism and its revenue optimality. We start with the central definition.

Definition 4.2.3. The single agent lottery pricing problems are *revenue linear* if $\mathrm{Rev}[\cdot]$ is linear. I.e., the optimal revenue for constraints $\hat{x} = \hat{x}^A + \hat{x}^B$ is $\mathrm{Rev}[\hat{x}] = \mathrm{Rev}[\hat{x}^A] + \mathrm{Rev}[\hat{x}^B]$.

Now consider the following two lower bounds on the optimal revenue for any allocation constraint $\hat{x}$. The constraint $\hat{x}$ is a monotone non-increasing function. As reverse step functions provide a basis for such functions, we can view $\hat{x}$ as a convex combination of reverse step functions. This convex combination can be sampled by drawing $\hat{q}$ at random from the distribution $G^{\hat{x}}$ with density $-\hat{x}'(q) = \frac{\mathrm{d}}{\mathrm{d}q}\hat{x}(q)$ and then posting price $V(\hat{q})$ (and allocation rule $\hat{x}^{\hat{q}}$). The allocation rule of the convex combination is exactly $\hat{x}$, its expected revenue lower bounds the optimal revenue subject to the constraint $\hat{x}$. A second approach is to use, instead of the posted pricing $V(\hat{q})$, the optimal lottery pricing for ex ante constraint $\hat{q}$. As the allocation rule for each of these mechanisms is weaker than the corresponding posted pricing allocation rule, the convex combination of the allocation rules (denote it by x) is weaker than the allocation constraint $\hat{x}$. Therefore, it is feasible for $\hat{x}$ and its revenue gives a lower bound on the optimal revenue for $\hat{x}$. Formally,

$$\begin{aligned} \mathrm{Rev}[\hat{x}] &\geq \mathbf{E}_{\hat{q}\sim G^{\hat{q}}}\left[-\hat{x}'(\hat{q})R(\hat{q})\right] \\ &= \left[-\hat{x}(\hat{q})R(\hat{q})\right]_0^1 + \mathbf{E}_q\left[R'(q)\hat{x}(q)\right] \\ &= \mathbf{E}_q\left[R'(q)\hat{x}(q)\right]. \end{aligned}$$

The second equality follows from integration by parts and the third equality from $R(1) = R(0) = 0$ (minor assumption: if we always serve or never serve the agent we obtain no revenue). This construction motivates the following definition.

Definition 4.2.4. The *marginal revenue* for an allocation constraint $\hat{x}$ is $\mathrm{MR}[\hat{x}] = \mathbf{E}_q[R'(q)\hat{x}(q)]$.

The definition of revenue linearity and the definition of the revenue curve (as the optimal revenue subject to the ex ante constraint $\hat{q}$) immediately imply the following theorem.

Theorem 4.2.1. *If the single-agent lottery pricing are revenue linear then the optimal revenue for an allocation constraint is equal to its marginal revenue, i.e., for all $\hat{x}$,* $\mathrm{Rev}[\hat{x}] = \mathrm{MR}[\hat{x}]$.

To show that the marginal revenue is equal to the optimal revenue for single-dimensional linear preferences, we must only prove revenue linearity. The proof of this theorem is a simple consequence of revenue equivalence and the simple facts that the optimal revenue for ex ante constraint $\hat{q}$ exceeds the posted pricing revenue from $V(\hat{q})$ but it has a weaker allocation rule; however, we defer it to Section A.1.

Theorem 4.2.2. *An agent with single-dimensional linear utility is revenue linear.*

Corollary 4.2.3. *For single-dimensional agents the optimal revenue for an allocation constraint is equal to its marginal revenue, i.e., for all $\hat{x}$,* $\mathrm{Rev}[\hat{x}] = \mathrm{MR}[\hat{x}]$.

Multi-agent Mechanisms Our single-agent discussion above is focused on optimizing revenue from a single-agent subject to an allocation constraint. We now look at the problem of optimizing expected revenue over agents in a multi-agent mechanism. The following is the standard argument from auction theory. For each agent, revenue is given by marginal revenue (Corollary 4.2.3). Relax incentive constraints (namely: monotonicity of the allocation rule) and optimize marginal revenue pointwise. Meaning, when the agent quantiles are $\boldsymbol{q} = (q_1, \ldots, q_n)$ select the allocation $\boldsymbol{x} = (x_1, \ldots, x_n)$ to maximize the cumulative marginal revenue $\sum_i R_i'(q_i) \cdot x_i$ subject to feasibility of $\boldsymbol{x}$

(e.g., for a single-item auction, serve the agent with the highest positive marginal revenue, or none if the marginal revenues are all negative). Now check that the incentive constraints hold: Notice that since revenue curves are concave, marginal revenues are monotone non-increasing, for any agent a stronger (lower) quantile corresponds to a weakly higher marginal revenue, so the intended allocation rule is monotone. Furthermore, as these allocations optimize marginal revenue pointwise for all profiles of agent quantiles, they certainly also maximize marginal revenue in expectation over the agent quantiles.

Comparing the above construction with the marginal revenue mechanism framework described in the introduction, the missing steps 1 and 4 are simple. For Step 1, the mapping from value to quantile is given by $V_i^{-1}(\cdot)$ for each agent i as described above. For Step 4, the appropriate payments can be calculated pointwise as follows: Agents that are not served pay nothing; an agent i that is served pays the value $V_i(\hat{q}_i)$ corresponding to her critical quantile $\hat{q}_i$, i.e., the quantile after which she would no longer served (via the payment identity).

Theorem 4.2.4. *The marginal revenue mechanism is revenue optimal for single-dimensional linear agents.*

4.3 Multi-dimensional and Nonlinear Preferences

In this section we start considering service-based environments that are much more general than single-dimensional, linear preferences that have been considered so far.

Bayesian mechanism design An agent has a private type t from type space T drawn from distribution F with density function f. In this paper we only consider settings where different agents' types are drawn independently. The agent may be assigned outcome w from outcome space W. This outcome encodes what kind of service the agent receives and any payments she must make for the service. In particular the payment specified by an outcome w is denoted by $\text{Payment}(w)$. The agent has a von Neumann–Morgenstern utility function: for type and deterministic outcome w her utility is $u(t, w)$, and when w is drawn from a distribution her utility is $\mathbf{E}_w[u(t, w)]$.[2] We will extend the definition of the utility function to distributions over outcomes $\Delta(W)$ linearly. For a random outcome w from a distribution, $\text{Payment}(w)$ will denote the expected payment.

There are n agents indexed $\{1, \ldots, n\}$ and each agent i may have her own distinct type space T_i, utility function u_i, etc. A *direct revelation* mechanism takes as its inputs a profile of types $\boldsymbol{t} = (t_1, \ldots, t_n)$, and then outputs for each agent i an outcome $\tilde{w}_i(\boldsymbol{t})$. The ex post outcome rule of the mechanism is $\tilde{w}_i : T_1 \times \cdots \times T_n \to \Delta(W_i)$. Agent i with type t_i, as the other agents' types are distributed over T_{-i}, faces an *interim outcome rule* $\tilde{w}_i(t_i)$ distributed as $\tilde{w}_i(t_i, \boldsymbol{t}_{-i})$ with $t_j \sim F_j$ for each $j \neq i$. We say that a mechanism is *Bayesian incentive compatible* if

$$u_i(t_i, \tilde{w}_i(t_i)) \geq u_i(t_i, \tilde{w}_i(t_i')), \quad \forall i, \forall t_i, t_i' \in T_i. \tag{BIC}$$

A mechanism is *interim individually rational* if

$$u_i(t_i, \tilde{w}_i(t_i)) \geq 0, \quad \forall i, \forall t_i \in T_i. \tag{IIR}$$

The mechanism designer seeks to optimize an objective subject to BIC, IIR, and ex post feasibility. We consider the objective of expected revenue, i.e., $\mathbf{E}_{\boldsymbol{t}}[\sum_i \text{Payment}(\tilde{w}_i(t_i))]$; however, any objective that separates linearly across the agents can be considered. Below we discuss the mechanism's feasibility constraint.

Service constrained environments In a *service constrained environment* the outcome w provided to an agent is distinguished as being a *service* or *non-service* outcome, respectively, with $\text{Alloc}(w) = 1$ or $\text{Alloc}(w) = 0$. There is a feasibility constraint restricting the set of agents that may be simultaneously served; there is no feasibility constraint

[2]This form of utility function allows for encoding of budgets and risk aversion; we do not require quasi-linearity.

on how an agent is served. With respect to the feasibility constraint any outcome $w \in W$ with $\text{Alloc}(w) = 1$ is the same. For example, payments are part of the outcome but are not constrained by the environment. An agent may have multi-dimensional and non-linear preferences over distinct service and non-service outcomes.

From least rich to most rich, standard service constrained environments are *single-unit environments* where at most one agent can be served, *multi-unit environments* where at most a fixed number of agents can be served, *matroid environments* where the set of agents served must be the independent set of a given matroid, *downward-closed environments* where the set of agents served can be specified by an arbitrary set systems for which subsets of a feasible set are feasible, and *general environments* where the feasible subsets of agents can be given by an arbitrary set system that may not even be downward closed.

Downward closure When optimizing revenue subject to a constraint on service probability, a downward-closed environment always allows the agent to be served with less often than the constraint desires. To provide a consistent framework that addresses non-downward closed environments as well, we will require equality of such constraints, but in downward-closed environments we will modify the outcome space to include each non-service outcome duplicated with the duplicate relabled as a service outcome. I.e., in a downward-closed environment as far as the feasibility constraint is concerned, we can always say that we served an agent when actually we did not. Such a modification to the outcome space implies that the revenue from constrained lottery pricing is always monotone in the constraint. Weaker constraints do not give lower revenue.

Revenue Curves The only aspect of the marginal revenue approach that translates identically from single-dimensional preferences to general preferences is the definition of the $\hat{q}$ ex ante optimal lottery pricing. This is the lottery pricing (i.e., collection of outcomes where the agent is permitted to choose her type-dependent favorite) denoted $\tilde{w}^{\hat{q}}(t)$ with the constraint that $\mathbf{E}_t[\text{Alloc}(\tilde{w}^{\hat{q}}(t))] = \hat{q}$ that optimizes revenue. For the optimal $\tilde{w}^{\hat{q}}(\cdot)$, the revenue curve for the agent is then given by $R(\hat{q}) = \mathbf{E}_t[\text{Payment}(\tilde{w}^{\hat{q}}(t))]$ as per Definition 4.2.1.

Allocation rules Our first challenge, then, in generalizing the marginal revenue approach to general preferences is that we cannot make an upfront transformation from the type space T of an agent to a $[0, 1]$ quantile space ordered by the strength of the agent. E.g., if the type is multi-dimensional then it is unclear which is stronger, a higher value in one dimension and lower in another or vice versa. In fact, which is stronger depends on the context, e.g., the competition from other agents.

Our approach is based on two observations. First, relative to a mechanism and for a particular agent, the relevant part of the mechanism is the (interim) outcome rule $\tilde{w}(\cdot)$. For a given outcome rule $\tilde{w}(\cdot)$ an ordering on types by strength can be defined. Simply, a type that is more likely to be served is stronger than a type that is less likely to be served. I.e., t is stronger than t' relative to $\tilde{w}(\cdot)$ if $\text{Alloc}(\tilde{w}(t)) \geq \text{Alloc}(\tilde{w}(t'))$. Second, (by the above mapping) any outcome rule $\tilde{w}(\cdot)$ induces an allocation rule $x(\cdot)$ that maps quantile to service probability. This allocation rule has a simple intuition in discrete type spaces: For each type $t \in T$ make a rectangle of width equal the probability of the type $f(t)$ and height equal to the service probability of the type $\text{Alloc}(\tilde{w}(t))$. Sort the types in decreasing order of height; the resulting piecewise constaint function from $[0, 1]$ to $[0, 1]$ is the allocation rule. This is generalized for continuous distributions as follows.

Definition 4.3.1. For an agent with $t \in T$ drawn from distribution F and outcome rule $\tilde{w}(\cdot)$, the *allocation rule* mapping quantiles to service probabilities is given by $x(\hat{q}) = \sup\{x : \mathbf{Pr}_{t\sim F}[\text{Alloc}(\tilde{w}(t)) \geq x]\}$.

Optimal Lottery Pricing With the definition of allocation rules for any lottery pricing in hand, allocation constrained lottery pricings generalize naturally. Even though the order on types may change from one lottery pricing to another, we can still ask for the lottery pricing with the optimal revenue subject to a constraint on its allocation rule. The optimal lottery pricing for allocation constraint $\hat{x}$ with cumulative allocation constraint $\hat{X}$ is given by the outcome rule $\tilde{w}(\cdot)$ that optimizes expected revenue subject to its corresponding allocation rule x with cumulative allocation rule X satisfying $X(\hat{q}) \leq \hat{X}(\hat{q})$ for $\hat{q} \in [0, 1]$ with equality at $\hat{q} = 1$. As per Definition 4.2.2 the optimal revenue for allocation constraint $\hat{x}$ is denoted $\mathrm{Rev}[\hat{x}]$.

We will generally denote by x the optimal allocation rule for constraint $\hat{x}$. The ex ante constraint on total service probability by $\hat{q}$ is given by the reverse step function at $\hat{q}$ denoted $\hat{x}^{\hat{q}}$; the corresponding allocation rule of the $\hat{q}$ optimal lottery pricing is denoted $x^{\hat{q}}$.

Revenue Linearity and Marginal Revenue Revenue linearity and marginal revenue have the same definitions (Definition 4.2.3 and Definition 4.2.4) as for single-dimensional preferences. The marginal revenue of an allocation constraint is $\mathrm{MR}[\hat{x}] = \mathbf{E}_q[R'(q)\hat{x}(q)]$. By its construction as the revenue of the appropriate convex combination of ex ante constrained mechanisms it is a lower bound on the optimal revenue, i.e., $\mathrm{Rev}[\hat{x}] \geq \mathrm{MR}[\hat{x}]$. Again by its construction, revenue linearity would imply it is equal to the optimal revenue.

There is nothing special about revenue curves for agents with general preferences over those for single-dimensional preferences. Given any revenue curve for a general agent, we can construct a single-dimensional agent with the exact same revenue curve. Theorem 4.2.4 shows that marginal revenue maximization is optimal for this single-dimensional analog.

Definition 4.3.2. The *optimal marginal revenue* for a service constrained environment with general agent preferences is the expected revenue (equal to marginal revenue) of the *single-dimensional analog* with each agent replaced by a single dimensional agent with the same revenue curve.

The framework thus defined affords two very natural questions. First, as for general preferences revenue may be strictly larger than marginal revenue, does the optimal marginal revenue approximate the optimal revenue? Second, as the implementation of the marginal revenue mechanism for single-dimensional preferences does not directly extend to general preferences (e.g., Steps 1 and 4), can we implement the marginal revenue mechanisms? In the remainder of this section we will focus on the revenue-linear special case, where the optimal revenue is the optimal marginal revenue, and we will answer the implementation question. Non-revenue-linear environments are considered in the next sections.

Implementation with Revenue Linearity We show now that the marginal revenue mechanism generalizes exactly for general preferences that satisfy revenue linearity. Moreover, in this case the marginal revenue mechanism inherits all of the nice properties of the marginal revenue mechanism for single-dimensional preferences. Namely, it deterministically selects the set of agents to serve, it is dominant strategy incentive compatible (truthful reporting is a best response for any actions of the other agents), and the mapping from types to quantiles to marginal revenues is deterministic and *context free*[3] in that it does not depend on the feasibility constraint or other agents in the mechanism. The mechanism, however, is optimal among the larger class of randomized and Bayesian incentive compatible mechanisms. As motivation for this result, we will show subsequently that there are multi-dimensional preferences that are revenue linear, e.g.,

[3]Note that this contrasts with recent algorithmic work in multi-dimensional optimal mechanism design where the optimal mechanism is characterized by mapping types stochastically to "virtual values" and this mapping is solved for from the feasibility constraint and the distributions of all agents types. See Alaei et al. [1] and Cai et al. [11, 12].

when multi-dimensional values are uniformly distributed on a hypercube.

The main challenge of implementing the marginal revenue mechanism is in specifying Step 1, i.e., the mapping from types to quantiles, and Step 4, i.e., selecting the appropriate outcomes for the set of agents that are served. If, however, each agent's types are orderable by the following definition, then both steps are essentially identical to the single-dimensional case.

Definition 4.3.3. A single-agent problem is *orderable* if there is an equivalence relation on the types, and there is an ordering on the equivalence classes, such that for any allocation constraint $\hat{x}$, the optimal outcome rule $\tilde{w}$ induces an allocation rule that is greedy by this ordering with ties between types in a same equivalence class broken uniformly at random.[4]

Orderability may look like a stringent and unlikely condition to hold generally. We note that it holds for single-dimensional agents and we show, more generally, it is a consequence of revenue linearity.

Theorem 4.3.1. *For any single-agent problem, revenue linearity implies orderability.*

The theorem is proved by the following two lemmas which characterize the structure of optimal lottery pricings; their proofs can be found in Section A.2.

Lemma 4.3.2. *For a revenue-linear single-agent problem, let x be the optimal allocation rule subject to some constraint $\hat{x}$. Then, for any $\hat{q}$ such that $R''(\hat{q}) \neq 0$ we have $X(\hat{q}) = \hat{X}(\hat{q})$.*

Lemma 4.3.2 in particular implies that for $\hat{q}$ with $R''(\hat{q}) \neq 0$ the $\hat{q}$-constrained lottery pricing (i.e., with allocation constraint given by the reverse step function $\hat{x}^{\hat{q}}$) has alloca-

[4]By greedy by the given ordering, we mean process each equivalance class in order and serve the corresponding types with as much probability as possible subject to the allocation constraint. As a consequence, given any prefix of the equivalance classes on types of measure $\hat{q}$ (according to the distribution on types), the allocation constraint imposed by $\hat{x}$ is binding. If all equivalance classes are measure zero, then resulting allocation rule is equal to the allocation constraint.

tion rule $x^{\hat{q}} = \hat{x}^{\hat{q}}$. I.e., the $\hat{q}$ lottery pricing has only full lotteries (all types are served with either probability one or zero).

For any such $\hat{q}$, define T_q to be the set of types allocated (with full lotteries) in the optimal allocation subject to $\hat{x}^{\hat{q}}$. The following lemma shows that these sets are nested.

Lemma 4.3.3. *For a revenue-linear single-agent problem, for any $\hat{q}_1 > \hat{q}_2$ and $R''(\hat{q}_1), R''(\hat{q}_2) \neq 0$, we must have $T_{\hat{q}_1} \supseteq T_{\hat{q}_2}$.*

Proof of Theorem 4.3.1. By Lemma 4.3.3, all $\hat{q}$ optimal lottery pricings order the types by the same equivalance classes. By revenue linearity the optimal lottery pricing for an allocation constraint $\hat{x}$ is a convex combination of the $\hat{q}$ optimal lottery pricings. Therefore, it allocates greedily to types by the same equivalance classes. □

Given orderability and the fact that (by Lemma 4.3.2) the optimal $\hat{q}$-constrained lottery pricings are full lotteries for $\hat{q}$ for which $R(\hat{q})$ is locally linear, the marginal revenue mechanism is easy to define.

Definition 4.3.4. The *marginal revenue mechanism* for orderable agents works as follows.

(a) Map reported types $\boldsymbol{t} = (t_1, \ldots, t_n)$ of agents to quantiles $\boldsymbol{q} = (q_1, \ldots, q_n)$ via the implied ordering.[5]

(b) Calculate the marginal revenue of each agent i as $R_i'(q_i)$.

(c) For each agent i, calculate the maximum quantile $\hat{q}_i$ that she could possess and be in the marginal revenue maximizing feasible set (breaking ties consistently).

(d) Offer each agent i the $\hat{q}_i$ optimal lottery pricing.

Proposition 4.3.4. *The marginal revenue mechanism deterministically selects a feasible set of agents to serve and is dominant strategy incentive compatible.*

[5]This ordering can be found by calculating the optimal single-agent mechanism for allocation constraint $\hat{x}(q) = 1 - q$.

Proof. Because ties are broken consistently, critical values cannot fall in intervals where the revenue curve is locally linear (and the marginal revenue curve is locally constant). Therefore, the lottery pricings offered to each agent are full lotteries; each type is deterministically served or not served. Feasibility follows as the set of agents that select service outcomes is exactly the marginal revenue maximizing set subject to feasibility. To verify the dominant strategy incentive compatibility consider any agent i's perspective. The parameter $\hat{q}_i$ is a function only of the other agents' reports; the agent's outcome is determined by the $\hat{q}_i$ optimal lottery which is incentive compatible for any $\hat{q}_i$. □

Proposition 4.3.5. *In service constrained environment with revenue-linear agents, the marginal revenue mechanism obtains the optimal marginal revenue (which equals the optimal revenue).*

A multi-dimensional revenue-linear example The example of the seller who can paint her car red or blue as she sells it to agents with independent and uniform values for each color is revenue linear (proof given in Section A.3). Therefore, the marginal revenue mechanism is optimal and its simple form can be derived from Definition 4.3.4 as follows. For an unit-demand agent with values for m variants of a service (i.e., possible colors of the car) distributed uniformly on $[0, 1]^m$, the quantile of each type $t = (t^1, \ldots, t^m)$ is $q = 1 - (\max_i t^i)^m$, i.e., the probability she has the maximum value over all kinds of services. The revenue function is $R(q) = q\sqrt[m]{1-q}$ if $q \leq \frac{m}{m+1}$, and $m(m+1)^{-(1/m+1)}$ otherwise. The marginal revenue function is $R'(q) = m(1-q)^{1/m-1}(m-(m+1)q)$ if $q \leq \frac{m}{m+1}$ and $R'(q) = 0$ otherwise. Notice that both the mapping and the marginal revenue function are monotone. Therefore serving the agent with the highest marginal revenue (Definition 4.3.4) means serving the player with the highest value for any kind of service and charging her the minimum she needs to bid to exceed the second-highest value (subject to the reserve of $\sqrt[m]{\frac{1}{m+1}}$ which is where the marginal revenue becomes zero).

4.4 Implementation

The marginal revenue mechanism (Definition 4.3.4) for agents with orderable types does not extend to general agents. In this section we give two approaches for defining the marginal revenue mechanism more generally. The first approach assumes that the parameterized family of $\hat{q}$-constrained lottery pricing satisfy a natural monotonicity requirement: that the probability that an agent with a given type is served is monotone in $\hat{q}$. Like the marginal revenue mechanism for orderable agents, this mechanism is dominant strategy incentive compatible. Unlike the marginal revenue mechanism for orderable agents, this mechanism does not deterministically select a set of agents to serve. The second approach is brute-force but easily computable and completely general. It results in a Bayesian incentive compatible mechanism. These two mechanisms will differ from the marginal revenue mechanism for orderable types only in the first (mapping types to quantiles) and last (serving each agent if her quantile is at most her critical quantile) steps; these changes can be mix-and-matched for different agents in the same mechanism.

Marginal revenue maximization for given revenue curves and feasibility constraint induces a profile of normalized interim allocation rules via the following simulation: Draw agent quantiles uniformly from $[0, 1]$; calculate the marginal revenues for each agent; serve the set of agents to maximize the marginal revenue. This simulation gives rise to the profile of normalized interim allocation rules that maximize marginal revenue in expectation. Denote these interim allocation rules by $\hat{x}_1^{MR}, \ldots, \hat{x}_n^{MR}$. Any real mechanism that maximizes marginal revenue should look to each agent i like sampling a $\hat{q}$-constrained lottery pricing with density $-\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\hat{x}_i^{MR}(\hat{q})$. The outcome rule of this convex combination for agent i is given by: $\tilde{w}_i^{MR}(t_i) = -\int_0^1 \tilde{w}^{\hat{q}}(t_i)\,\mathrm{d}\hat{x}_i^{MR}(\hat{q})$ where $\tilde{w}^{\hat{q}}$ is the outcome rule for the $\hat{q}$-step mechanism. Our goal in this section is to find the multi-agent mechanism that induces these interim outcome rules $\tilde{w}_1^{MR}, \ldots, \tilde{w}_n^{MR}$ and is ex post feasible.

We conclude this section by describing a relevant class of agents for which the ex ante constrained lottery pricing satisfy the monotonicity property required by the first approach of this section. The example is one of a single-dimensional agent with a public budget.

4.4.1 Monotone Ex Ante Constrained Lottery Pricings

We define below a simple extension of the marginal revenue mechanism for orderable types for the case where the single-agent step mechanisms satisfy a natural monotonicity property. The resulting mechanism is based on a randomized mapping from types to quantiles that is independent across the agents.

Definition 4.4.1. An agent has *monotone ex ante constrained lottery pricing* if, given her type, the probability she wins in the $\hat{q}$-constrained lottery pricing $\mathcal{M}^q$ is monotone non-decreasing in $\hat{q}$.

Suppose that the $\hat{q}$-constrained lottery pricing $\mathcal{M}^q$ for an agent each consist of a menu of full lotteries. I.e., for any type of the agent she will choose a lottery that either serves her with probability 1 or zero. In this case the monotone ex ante constrained lottery pricing assumption would require that the sets of types served for each $\hat{q}$ are nested. There is a simple deterministic mapping from types to quantiles in this case: set the quantile of a type to be the minimum $\hat{q}$ such that the $\hat{q}$-constrained lottery pricing serves the type. Below, we generalize this selection procedure to the case of partial lotteries (where types may be probabilistically served or not).

Recall that the $\hat{q}$-constrained lottery pricing $\mathcal{M}^q$ has allocation rule $\tilde{x}^{\hat{q}}$ that maps types to probability of service. Fix the type of the agent as t and consider the function $G_t(q) = \tilde{x}^{\hat{q}}(t)$ which, by the monotonicity condition above, can be interpreted as a cumulative distribution function. Notice that $\mathcal{M}^q$ has ex ante probability of service $\mathbf{E}_t[\tilde{x}^{\hat{q}}(t)] = \hat{q}$. Therefore, if t is drawn from the type distribution and then q is drawn from G_t then the distribution of q is uniform on $[0, 1]$.

Lemma 4.4.1. *If $t \sim F$ and $q \sim G_t$ then q is $U[0, 1]$.*

Definition 4.4.2. The *marginal revenue mechanism* for agents with monotone step mechanisms works as follows.

(a) Map reported types $\boldsymbol{t} = (t_1, \ldots, t_n)$ of agents to quantiles $\boldsymbol{q} = (q_1, \ldots, q_n)$ by sampling q_i from the distribution with cumulative distribution function $G_{t_i}(q) = \tilde{x}_i^{\hat{q}}(t_i)$.

(b) Calculate the marginal revenue of each agent i as $R_i'(q_i)$.

(c) For each agent i, calculate the maximum quantile q_i^* that she could possess and be in the marginal revenue maximizing feasible set (breaking ties consistently).

(d) For each agent i, offer the mechanism $\mathcal{M}_i^{\hat{q}_i^*}$ conditioned so that i is served if $q_i \leq q_i^*$ and not served otherwise.

The last step of the marginal revenue mechanism warrants an explanation. In the $\hat{q}_i^*$-constrained lottery pricing $\mathcal{M}_i^{\hat{q}_i^*}$, the outcome that i would obtain with type t_i may be a partial lottery, i.e., it may probabilistically serve i or not. The probability that i is served is $\tilde{x}_i^{\hat{q}_i^*}(t_i) = \mathbf{Pr}_{q_i}[q_i \leq q_i^*]$ by our choice of q_i. When we offer agent i the mechanism $\mathcal{M}_i^{\hat{q}_i^*}$ we must draw an outcome from the distribution given by $\tilde{w}_i^{\hat{q}_i^*}(t_i)$. Some of these outcomes are service outcomes, some of these are non-service outcomes. If $q_i \leq q_i^*$ then we draw an outcome from the distribution $\tilde{w}_i^{\hat{q}_i^*}(t_i)$ conditioned on service; if $q_i > q_i^*$ then we draw an outcome conditioned on no-service. Notice that it may not be feasible to serve all agents who receive non-trivial partial lottery. This method coordinates across the partial lotteries which agents to serve to maintain the right distribution on agent outcomes and ensure feasibility.

Proposition 4.4.2. *The marginal revenue mechanism for agents with monotone step mechanisms is feasible and dominant strategy incentive compatible.*

Proof. Feasibility follows as the set of agents that select service outcomes is exactly the marginal revenue maximizing set subject to feasibility. To verify the dominant strategy incentive compatibility consider any agent i's perspective. The parameter q_i^* is a function only of the other agents' reports; the agent's outcome is determined by the $\hat{q}_i^*$-constrained lottery pricing $\mathcal{M}_i^{\hat{q}_i^*}$ which is incentive compatible for any q_i^*. $\square$

Theorem 4.4.3. *The marginal revenue mechanism for agents with monotone step mechanisms implements marginal revenue maximization.*

Proof. From each agent i's perspective, the other agents' quantiles are distributed independently and uniformly on $[0, 1]$ (Lemma 4.4.1). Therefore, this agent faces a distribution over ex ante constrained lottery pricing that is identical to the distribution of "critical quantiles" in the maximization of marginal revenue, i.e., with density $-\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\hat{x}^{MR}(\hat{q})$. $\square$

4.4.2 General Ex Ante Constrained Lottery Pricings

For general agents for whom the ex ante constrained lottery pricings do not satisfy the monotonicity condition (Definition 4.4.1), we give in Section A.4 an efficiently implementable procedure to extract the optimal marginal revenue (recall Definition 4.3.2). The key to the proof of Theorem 4.4.4 is a variation of the technique of vector majorization [30].

Theorem 4.4.4. *For downward-closed service constrained environments, the optimal marginal revenue can be efficiently extracted by a Bayesian incentive compatible marginal revenue mechanism (see Definition A.4.2).*

4.5 Approximation

In previous sections, we have shown that for any collection of agents the marginal revenue mechanism can be implemented. We know that for context-free agents, this extracts the optimal revenue. In this section, we show that this revenue is a good approximation to the optimal revenue quite generally.

We will give two approaches for approximation bounds. The first kind of bound is based on the single-agent problem, i.e., the distribution and type space. If we can show that for all allocation constraints, the marginal revenue is a good approximation to the optimal revenue, then the marginal revenue mechanism is a good approximation to the optimal mechanism. The second approach will derive bounds from the feasibility constraint. Clearly, with no feasibility constraint, marginal revenue maximization is optimal. We will show that for matroid environments, it gives a $1 - 1/e$ approximation, and for general downward-closed environments, it gives a $O(\log n)$ approximation.

Of course, if we are in an environment where our agent-based arguments imply an α-approximation and our feasibility-based arguments imply a β-approximation, the marginal revenue mechanism is in fact a $\min(\alpha, \beta)$-approximation. In context-free environments $\alpha = 1$ (and marginal revenue is optimal); the approximation smoothly degrades in α as the environment becomes less revenue linear until it reaches the approximation bound β given by the feasibility constraint.

4.5.1 Agent-based Approximation

If, for all allocation rules, the marginal revenue is close to the optimal revenue, then marginal revenue maximization is approximately optimal. One approach to deriving such a bound is to give a linear upper bound on the optimal revenue and a lower bound through a class of pseudo ex ante constrained lottery pricings. A pseudo step mechanism respects a step constraint but may not be optimal. If for every quantile $\hat{q}$ the pseudo $\hat{q}$-constrained lottery pricing approximates the linear upper bound, then marginal revenue maximization approximates the optimal revenue for all allocation constraints. Furthermore, these pseudo ex ante constrained lottery pricings can be directly optimized over and the same approximation factor is obtained. Such an approach might be desirable if the pseudo ex ante constrained lottery pricings are better-behaved than the (optimal) ex ante constrained lottery pricings, e.g., if they are easy to compute, respect an ordering on types, or are step monotone.

This approach is formalized by the following sequence of definitions and propositions.

Proposition 4.5.1. *If for any agent i and allocation rule x_i, the marginal revenue* $\mathrm{MR}(x_i)$ *is at least an α fraction of the optimal revenue* $\mathrm{Rev}(x_i)$*, then the marginal revenue mechanism in the multi-agent setting is an α-approximation to the optimal mechanism.*

Definition 4.5.1. A *linear revenue bound*, UB, is a function mapping an allocation constraint to a revenue, which is

- linear in the allocation constraint, i.e., for all allocation constraints $\hat{x}_a$ and $\hat{x}_b$ and $\gamma \in [0, 1]$, $\mathrm{UB}(\gamma \hat{x}_a + (1 - \gamma)\hat{x}_b) = \gamma\, \mathrm{UB}(\hat{x}_a) + (1 - \gamma)\, \mathrm{UB}(\hat{x}_b)$.
- an upper bound on revenue for all allocation constraints, i.e., $\forall \hat{x},\ \mathrm{UB}(\hat{x}) \geq \mathrm{Rev}(\hat{x})$, and

Definition 4.5.2. A *pseudo ex ante constrained lottery pricing* is one that respects a step constraint but is not necessarily revenue optimal for such a constraint. The revenue of the pseudo $\hat{q}$-constrained lottery pricing $\tilde{\mathcal{M}}^q$ is denoted $\tilde{R}(\hat{q})$; and the *pseudo marginal revenue* for allocation constraint $\hat{x}$ is $\mathrm{PMR}(\hat{x}) = \mathbf{E}[\tilde{R}'(\hat{q})\hat{x}(\hat{q})]$.

We can assume without loss of generality that the pseudo marginal revenue $\tilde{R}$ is concave. If it is not we could always redefine the class by taking its closure with respect to convex combination and letting the pseudo $\hat{q}$-constrained lottery pricing be the revenue-optimal mechanism in the class that serves with ex ante probability $\hat{q}$.

Proposition 4.5.2. *For a given linear revenue bound* UB*, if for all $\hat{q}$ the pseudo $\hat{q}$-constrained lottery pricing α-approximates the bound on the $\hat{q}$-step constraint* $\mathrm{UB}(\hat{x}^{\hat{q}})$*, then the pseudo marginal revenue α-approximates the optimal revenue for all allocation constraints.*

Proof. This proposition follows from linearity of both the revenue bound and pseudo marginal revenue. □

Definition 4.5.3. The *pseudo marginal revenue mechanism* is the one that maximizes pseudo marginal revenue via any of the approaches of Definition 4.3.4, Definition 4.4.2, or Definition A.4.2 that applies.

Pseudo ex ante constrained lottery pricing for downward-closed unit-demand agents For clarity with respect to the literature we will refer to this agent as being unit-demand and the types of services as being distinct items. Without an allocation constraint, this problem has seen extensive studied. Briest et al. [8] show that the optimal lottery pricing can be calculated by a linear program that has size equal to the number of distinct types of the agents. When the agent's value for the items are independently distributed, Cai and Daskalakis [10] give a dynamic program for approximating the optimal item pricing to within a $(1 + \epsilon)$ factor for any ϵ in time polynomial in the number of items. These results are distinct in two ways. First, the first result is optimal with respect to randomized mechanisms whereas the second is (nearly) optimal with respect to deterministic mechanisms. Second, the first result would require time exponential in the number of items for a product distribution, while the second result is polynomial (but requires a product distribution). It is not known whether the optimal lottery pricing for product distributions can be calculated arbitrarily closely in polynomial time in the number of items. (Recently, Daskalakis et al. [20] showed that it is #P-hard to calculate the *exactly optimal* lottery pricing in this scenario.) Attempting to address this question, a combination of the work of Chawla et al. [16, 15] shows that for product distributions, item pricing is a 4-approximation to lottery pricing. Furthermore, there is an item pricing that is very simple to describe that satisfies this bound. We generalize this theory to single-agent problems with a supply constraint $\hat{x}$ and use it to give a 4-approximate class of pseudo ex ante constrained lottery pricing for unit-demand, downward-closed agents.

Consider the syntactically-related problem of selling a single item to one of m single-dimensional agents with values drawn from a product distribution, i.e., the value v_i of agent i is drawn independently from F_i. As described earlier, the optimal auction for this single-dimensional problem is well understood. Agent values are mapped to virtual values (equivalent to each agent's marginal revenue), and the agent with the highest positive virtual value is selected as the winner of the auction. We refer to this auction environment as the single-dimensional *representative environment*, the revenue obtained by the optimal auction as the *optimal representative revenue,* and the agents participating in the auction as *representatives*.

Notice that if these representatives were all colluding together the problem would be identical to our original single-agent unit-demand problem. We refer to this environment as the *unit-demand environment* and the revenue of the optimal lottery pricing as the *optimal unit-demand revenue*. The approach of Chawla et al. [16] is to try to mimic the outcome of the optimal auction for the representative environment to obtain an approximately optimal pricing in the unit-demand environment. As the optimal auction in the representative environment orders representatives by virtual value, a natural approach to pricing the items in the unit-demand environment is to set a uniform virtual price, i.e., the price for each item has the same virtual value (with respect to the distribution from which the agent's value for that item is drawn).[6] Chawla et al. [16] show that the unit-demand revenue of such a pricing is a 2-approximation to the optimal representative revenue; Chawla et al. [15] show that the optimal unit-demand revenue (e.g., from lottery pricings) is at most twice the optimal representative revenue. Combining these two results, *uniform virtual pricing* is a 4-approximation to the optimal unit-demand revenue.

We generalize the approach above to single-agent problem of serving an agent with independent values for m items subject to an allocation constraint $\hat{x}$. In particular, twice the optimal representative revenue is a linear revenue bound (Definition 4.5.1),and we define a class of pseudo ex ante constrained lottery pricing where the pseudo $\hat{q}$-constrained lottery pricing is given by a uniform virtual pricing that sells with probability $\hat{q}$. Since the virtual values are weakly increasing in the representative agents' values, the sets of types served by these pseudo ex ante constrained lottery pricing are nested. Therefore, the pseudo marginal revenue mechanism can be implemented via the marginal revenue mechanism for orderable agents (Definition 4.3.4). Finally, we show that for all $\hat{q}$ the pseudo $\hat{q}$-constrained lottery pricing is a 4-approximation to the linear upper bound given by twice the optimal representative revenue. This result with Proposition 4.5.2 implies that the pseudo marginal revenue mechanism is a 4-approximation to the optimal revenue. The proof of Theorem 4.5.3, below, is a rather straightforward extension of Chawla et al. [16, 15] and we include it in Section A.5.

Definition 4.5.4. The pseudo $\hat{q}$-constrained lottery pricing for a unit-demand agent with values for items drawn independently from $F_1, \ldots, F_n$ is given by the pricing that sets a uniform virtual price for the items such that the probability that the agent buys any item is equal to $\hat{q}$. (If this class does not have a monotone non-decreasing pseudo revenue

[6] As mentioned above, a representative's virtual value is equal to their marginal revenue. For clarity of discussion and to disambiguate the marginal revenue of the unit demand agent versus that of her representatives we will refer to the representative marginal revenue as virtual value.

curve $\tilde{R}(\cdot)$ we invoke downward closure to make it monotone; if this class does not have a concave pseudo revenue curve we take its closure with respect to convex combination to make it concave.)

Theorem 4.5.3. *In downward-closed (service constrained) environments with unit-demand agents, both the pseudo marginal revenue mechanism and the marginal revenue mechanisms are a 4-approximation to the optimal revenue.*

4.5.2 Feasibility-based Approximation

We now show that the feasibility constraint implies an approximation bound as well. As a first simple bound, if there is no feasibility constraint (e.g., for digital goods) then marginal revenue maximization is optimal. Below we give a $e/(e-1)$ approximation bound for matroid environments, and, for n linear-utility agents, an $O(\log n)$ bound for downward-closed environments.

Matroid environments Marginal revenue maximization is an $e/(e-1)$ approximation when the feasibility constraint is induced by independent sets of a matroid set system. This same approximation factor governs single-item auctions. For k-unit environments we obtain a $1/(1-(2\pi k)^{-1/2})$-approximation. These results follow from the correlation gap approach of Yan [56].

Theorem 4.5.4. *In a matroid environment the optimal marginal revenue is a $e/(e-1)$-approximation to the optimal revenue; for k-unit environments it is a $1/(1-(2\pi k)^{-1/2})$-approximation.*

Proof sketch. Suppose the optimal mechanism serves agent i with ex ante probability $\hat{q}_i^*$. Relax the feasibility constraints and consider maximizing revenue subject to ex ante probability $\hat{q}_i^*$ for each agent i. This revenue is only greater and it is precisely

$\sum_i R_i(\hat{q}_i^*)$. Sort agents by $R_i(\hat{q}_i^*)/\hat{q}_i^*$, i.e., bang-per-buck, and run the greedy matroid algorithm: if it is possible to serve i when i is visited by the algorithm, then offer i the $\hat{q}_i^*$-constrained lottery pricing $\mathcal{M}^{\hat{q}_i^*}$ (by definition, the optimal mechanism that serves with ex ante probability $\hat{q}_i^*$). It follows immediately from the main theorem of Yan [56] is that this is an $e/(e-1)$-approximation for general matroids and an $1/(1-(2\pi k)^{-1/2})$-approximation for k-uniform matroids. This greedy-based mechanism's revenue is given by its marginal revenue, and therefore the marginal revenue maximizer is only better. $\square$

Downward-closed environments In this section we show that in downward-closed environments where the agents have linear utilities, the optimal marginal revenue is a logarithmic approximation, in the number of agents, to the optimal revenue.

The intuition for the proof is as follows. If we consider allocation constraints with a minimum probability of allocating to any type of 2^{-K}, then the allocation constraint can be partitioned into K pieces with the highest and lowest probability of allocation within each piece being within a factor of two of each other. The revenue of each piece can be approximated by a $\hat{q}$-constraint scaled appropriately so that it is dominated by the original allocation constraint. (In order for the revenue of the $\hat{q}$-constraint lottery pricing to scale appropriately, we need the agent to have linear utility.) The total revenue is then at most an $O(K)$-fraction of the revenue of the best such scaled step constraint. We will be able to restrict attention to allocation constraints with $K \approx \log n$.

We start by proving the following lemma.

Lemma 4.5.5. *For an agent with linear utility, any allocation constraint with minimum probability $\hat{x}(1) \geq 2^{-K}$ has revenue at most $2K\,\mathrm{MR}(\hat{x})$.*

Proof. Let $R^* = \mathrm{Rev}(\hat{x})$ be the optimal revenue for allocation constraint $\hat{x}$. Define sequence of quantiles $0 = q_0 \leq q_1 \leq \cdots \leq q_K = 1$ such that $x(q_{j-1}) \leq 2x(q_j)$. Define R_j^* to be the expected revenue from types that are mapped to a quantile in $[q_{j-1}, q_j]$, where quantile of a type is the probability that a type drawn at random has a higher probability of service. Therefore, the revenue of the mechanism is $R^* = \sum_{j=1}^{K} R_j^*$. Then there must exist j^* such that $R^* \leq KR_{j^*}^*$. In what follows, we define normalized allocation rules $z_j(\cdot)$ for all j, such that $z_j \leq x$, and also $R_j^* \leq 2\,\mathrm{MR}(z_j)$. In particular, for j^* we will have $z_{j^*} \leq x$, and $2\,\mathrm{MR}(z_{j^*}) \geq R_{j^*}^* \geq R^*/K$. This implies that

$$2 \max_{z \leq x} \mathrm{MR}(z) \geq 2\,\mathrm{MR}(z_{j^*}) \geq R^*/K.$$

Define function $z_j(\cdot)$ to be $z_j(q) = x(q_{i+1})$ if $q \leq q_{i+1} - q_i$, and 0 otherwise. Notice that for any q, we have $z_j(q) \leq x(q)$, and therefore $z_j \leq x$, by the definition of dominance in downward-closed environments.

We now show that for z_j defined above, $R_j^* \leq 2\,\mathrm{MR}(z_j)$. By construction of z_j, and recalling that $x(q_j) \leq 2x(q_{j+1})$,

$$\begin{aligned} 2\,\mathrm{MR}(z_j) &= 2\int_0^1 z_j(q) R'(q)\mathrm{d}q \\ &= 2x(q_{j+1})R(q_{j+1} - q_j) \\ &\geq x(q_j)R(q_{j+1} - q_j) \end{aligned}$$

It is therefore sufficient to show that $x(q_j)R(q_{j+1} - q_j) \geq R_j^*$. Recall that R_j^* is the revenue from types that are mapped to quantiles in $[q_j, q_{j+1}]$. Any type in $[q_j, q_{j+1}]$ is allocated in x with probability at most $x(q_j)$. Now define the set of lotteries L to be the lotteries chosen by types in $[q_j, q_{j+1}]$, and offer them to the agent [7]. Notice that types in $[q_j, q_{j+1}]$ choose the same lottery in L as they did in x. As a result, the measure of the types that choose some lottery in L is at least $q_{j+1} - q_j$. Now remove lotteries from L, from the one with lowest price, until the measure of types that choose some lottery is exactly $q_{j+1} - q_j$. Call this new set of lotteries L'. Notice that the revenue from L' is at least R_j^*. Now recall that all the lotteries in L, and therefore L', allocate with probability

[7] By the taxation principle, any incentive compatible mechanism consists of a set of lotteries, from which the agent chooses the one maximizing her utility.

at most $x(q_j)$. So it is feasible to define a new set of lotteries L'' to be the lotteries in L' scaled up by $1/x(q_j)$ [8] As the agent has linear utility, from L'' each type would choose her favorite lottery in L' scaled up by $1/x(q_j)$. The revenue from L'' is therefore at least $R^*_j/x(q_j)$. Since a fraction $q_{j+1} - q_j$ of types buy some lottery in L'', by definition the revenue that we get is at most $R(q_{j+1} - q_j)$. We conclude that $R^*_j/x(q_j) \leq R(q_{j+1} - q_j)$.

To complete the proof, recall that for downward-closed environments revenue curves are monotone non-decreasing so marginal revenues are non-negative. Therefore, by the definition of marginal revenue and dominance, $\mathrm{MR}(\hat{x}) \geq \mathrm{MR}(z_j)$ for all j. □

Theorem 4.5.6. *In downward-closed environments with n linear-utility agents, the optimal marginal revenue is a* $4 \log n$*-approximation to the optimal revenue.*

Poorf of Theorem 4.5.6. Consider an alternative mechanism that runs the optimal mechanism with probability $1/2$, and otherwise picks an agent at random and outputs an arbitrary outcome that services that agent, regardless of his type and without charging him. The revenue of the alternative mechanism is half the revenue of the optimal revenue. Let $x_1, \ldots, x_n$ be the allocation rules for the alternative mechanism. Notice also that by construction of the alternative mechanism, for each i and $q \in [0, 1]$ we have $x_i(q) \geq 1/2n$. Therefore we can invoke Lemma 4.5.5 with $K = \log 2n$ to conculde that that the revenue of the alternative mechanism is at most

$$2 \log n \sum\nolimits_i \mathrm{MR}_i(x_i).$$ □

[8] A lottery is said to be scaled up by a factor of α if its price and the probability of each of the service offered is multiplied by α.

4.6 Extending Techniques From Single-Dimensional Linear Utility Settings

The marginal revenue approach allows natural generalizations of techniques developed in the single-dimensional settings. These techniques include simple reserve price based mechanisms and prior-independent mechanisms. Theorem 4.5.3, for example, can be naturally combined with such techniques.

Example 4.6.1. For the environment of linear-utility unit demand agents, if, for each agent, her valuation for each item is i.i.d. drawn from a regular distribution (whereas different agents' distributions are allowed to be different), the pseudo ex ante contrained lottery pricings (Definition 4.5.4) for each agent is a fixed price on all items, allowing the agent to choose her favorite one. Any mechanism, certainly including the pseudo marginal revenue mechanism, that is the convex combination of such lottery pricings can effectively reduce the type of an agent to the highest value she holds for any item. This allows the following generalization of results from the single-dimension literature:

Simple vs. Optimal Combining Theorem 4.5.3 and techniques in Hartline and Roughgarden [32], in matroid settings, the revenue of the VCG mechanism with an anonymous reserve price on all services is a 16-approximation to the optimal revenue; if the reserve prices for each agent are set to the optimal prices used in a single-agent pricing problem, the VCG mechanism is an 8-approximation.

Prior-Independent Mechanisms via Single Sampling Combining Theorem 4.5.3 and techniques in Dhangwatnotai et al. [21], in matroid settings, if for each agent there is κ other ones with the same valuation distribution, then in the VCG auction one can randomly choose one of them, using her highest value among the services as a reserve price for the others, to obtain a prior-independent (since no prior information is used in this mechanism) $8 \cdot \frac{\kappa}{\kappa-1}$-approximation to the optimal revenue.

As another example, for agents whose valuations for different services are drawn independently from (not necessarily identical) regular distributions, one can define another set of pseudo ex ante constrained lottery pricing differently from Definition 4.5.4, wherein for each $\hat{q} \in [0, 1]$, there is a uniform price on all services and the agent may choose whichever she likes. Using techniques similar to those in Section A.5, one can show that this set of lottery prices 8-approximates the linear upper bound developed in Lemma A.5.1. This again allows one to reduce the type of an agent to her highest value among all services, and the generalizations listed in Example 4.6.1 translate easily. In particular, we obtain:

Theorem 4.6.1. *In multi-service linear utility environments, when each agents' values for the services are drawn independently from regular distributions, the VCG mechanism with an anonymous reserve price on all services is a* 32*-approximation to the optimal revenue.*

CHAPTER 5

SIMULTAENOUS ITEM AUCTIONS

5.1 Introduction

In previous chapters we have mainly considered the design of revenue-optimal, incentive compatible mechanisms. Another central problem in algorithmic mechanism design is to determine how best to allocate resources among individuals to maximize *social welfare*. Much of the theoretical work in this field to date has focused on solving such problems also via incentive compatible mechanisms. Such an approach has theoretical appeal, but incentive compatible mechanisms with good welfare warranties, e.g. the VCG auctions, tend to be complex and are rarely used in practice. Instead, it is common to forego incentive compatibility and use simpler mechanisms. Canonical examples of such auctions are the generalized second price (GSP) auctions for online advertising [23, 53], and the ascending price auction for electromagnetic spectrum allocation [42]. Given that such simple auctions are used in practice, it is of crucial importance to determine how they actually perform when used by rational agents.

Consider the problem of resolving a *combinatorial auction*. In such a problem there is a large set M of m objects for sale, and n potential buyers. Each buyer has a private value function $v_i \colon 2^M \to \mathbb{R}_{\geq 0}$ mapping sets of objects to their associated values. The goal of the market designer is to decide how to allocate the objects among the buyers to maximize the overall social efficiency. One approach would be to elicit the valuation function from each bidder, then attempt to solve the resulting optimization problem. However, the valuation function is an object of exponential size, and this approach leads inevitably to large communication and computation complexity overheads. It is not surprising, therefore, that in existing online marketplaces such as eBay, buyers do not express their (potentially complex) preferences directly; rather, each item is auctioned independently, and a buyer is forced to bid separately on individual items. This approach is simple and natural, and relieves the burden of expressing a potentially complex valuation function. On the other hand, this limited expressiveness could potentially lead to inefficient outcomes. This begs the question: how well does the outcome of simultaneous item auctions approximate the socially optimal allocation?

In order to evaluate the performance of non-truthful mechanisms, we take the economic viewpoint that self-interested agents will apply bidding strategies at equilibrium, so that no agent can unilaterally improve his outcome by changing his strategy. We apply a quantitative approach, and ask how well the performance at equilibrium approximates the socially optimal outcome. Since there may potentially be multiple equilibria, we will bound the performance in the worst case over equilibria. Put another way, our approach is to use the *price of anarchy* as a performance measure for the analysis of mechanisms.

The fact that equilibria of simultaneous auctions might not be socially optimal was first observed by Bikhchandani [7], who studied the complete information[1] setting. As he states:

> "Simultaneous sealed bid auctions are likely to be inefficient under complete information and hence, also under the more realistic assumption of incomplete information about buyer reservation values."

Our goal is to bound the extent of this inefficiency in the incomplete information setting. To this end, we model incomplete information using the standard Bayesian framework, as we have done in previous chapters. Again we will assume that the agents' valuations/types are *independently* drawn from distributions. This product distribution is commonly known to all of the participants; we think of this as representing the public's aggregate beliefs about the buyers in the market. While the distributions are common

[1] In a complete (or full) information setting, it is assumed that the bidders' valuations are commonly known to all participants

knowledge, each agent's true valuation is private. This Bayesian model generalizes the full-information model of Nash equilibrium, which implicitly supposes that the type profile is known by all participants. Note that while the agents are aware of the type distribution, the mechanism (which applies simultaneous item auctions) is prior-free and hence agnostic to this information.

Pricing and Efficiency in Simultaneous Auctions We consider separately the case in which items are sold via first-price auctions (in which the player who bids highest wins and pays his bid), and the case of second-price auctions[2] (in which the winning bidder pays the second-highest bid). The differences between first and second-price simultaneous auctions have received significant attention in the recent literature. For example, a pure Nash equilibrium of our mechanism with simultaneous first-price auctions is equivalent to a Walrasian equilibrium [7, 34], and therefore must obtain the optimal social welfare [40]. On the other hand, every pure Nash equilibrium for simultaneous second-price auctions is equivalent to a Conditional equilibrium, and hence obtains at least half of the optimal social welfare [27]. While these constant factor bounds are appealing, their power is marred by the fact that pure equilibria do not exist in general. In fact, based on the equivalence results above, their existence is quite restrictive (e.g., for simultaneous first-price auctions, existence is guaranteed for an extremely restrictive family of valuations, called *gross substitutes* [29]). Moreover, pure Nash equilibria rely on the very strong assumption of full information, which is rare in practice.

Can we hope for such constant-factor bounds to hold for general Bayes-Nash equilibria? For general valuations the answer is no. Consider, for example, the case of a buyer who has a very large value for the set of all objects for sale, but no value for any strict subset. In this case, any positive bid carries great risk: the buyer might win

[2]Second-price item auctions are also known as Vickrey auctions; we will use these terms interchangeably.

some items but not others, leaving him with negative utility. It therefore seems that complements do not synergize well with item bidding, and indeed it has been shown by Hassidim et al. [34] that the price of anarchy (with respect to mixed equilibria) in a first-price auction can be as high as $\Omega(\sqrt{m})$ when bidders' valuations exhibit complementarities. The same lower bound can be easily extended to the case of second-price auctions.[3]

Our main result in this chapter is that *the presence of complements is the only barrier to a constant price of anarchy*. We show that when buyer valuations are complement-free (a.k.a. subadditive), the (Bayesian) price of anarchy of the simultaneous item auction mechanism is at most a constant, in both the first- and second-price auctions.

For first-price auctions, we show that any Bayes-Nash equilibrium yields at least half of the optimal social welfare. This improves upon the previously best-known bound of $O(\log n)$ due to Hassidim et al. [34], where n is the number of bidders.

Result 1: [**BPoA$\leq$ 2 in simultaneous first-price auctions.**] *When buyers have subadditive valuations, the Bayesian price of anarchy of the simultaneous first-price item auction mechanism is at most* 2.

For simultaneous Vickrey auctions, it is not possible to bound the worst-case performance at equilibrium, even when there is only a single object for sale. This impossibility is due to arguably unnatural equilibria in which certain players grossly overreport their values, prompting others to bid nothing. To circumvent this issue one must impose an assumption that agents avoid such "overbidding" strategies. In the *strong no-overbidding assumption*, used by Christodoulou et al. [17] and Bhawalkar and Roughgarden [5], it is assumed that each agent i chooses bids so that, for every set of objects S, the sum of the bids on S is at most $v_i(S)$. We show that under this assumption, the Bayesian

[3] As explained in the sequel, to obtain meaningful results in second-price auctions one needs to impose *no-overbidding* assumptions on the bidding strategies, defined formally in Section 5.2.3. The $\Omega(\sqrt{m})$ lower bound extends to the case of second-price auctions under the *weak no-overbidding* assumption. The alternative *strong no-overbidding* assumption is meaningless in the case of complements, as it precludes item bidding altogether.

price of anarchy for simultaneous Vickrey auctions is at most 4. This improves upon the previously best-known bound of $O(\log n)$ due to Bhawalkar and Roughgarden [5].

Result 2: [**BPoA≤ 4 in simultaneous second-price auctions.**] *When buyers have subadditive valuations, the Bayesian price of anarchy of the simultaneous Vickrey auction mechanism is at most* 4*, under the strong no-overbidding assumption.*

The strong no-overbidding assumption is quite strong, as it must hold for *every* set of items. A somewhat weaker assumption, referred to as *weak no-overbidding*, requires that the no overbidding condition holds only in expectation over the distribution of sets won by a player at equilibrium. That is, agents are said to be *weakly no-overbidding* if they apply strategies such that expected value of each agent's winnings is at least the expected sum of his winning bids [27]. Roughly speaking, weak no-overbidding supposes that agents are generally averse to winning sets with bids that are higher than their true values. However, unlike strong no-overbidding, it does not preclude strategies in which an agent overbids on sets that he does not expect to win, i.e. in order to more accurately express his willingness to pay for other sets.

Notably, the BNE outcomes under the two no-overbidding assumptions are incomparable; while the weak assumption is more permissive, and thus enables a richer set of behaviors in equilibrium, it also introduces new ways to deviate from the prescribed equilibrium. Therefore, a constant bound on the Bayesian PoA under the weakly no-overbidding assumption does not follow directly. Nevertheless, we show that the bound of 4 on the Bayesian PoA extends also to the case of weakly no-overbidding agents.

Bhawalkar and Roughgarden [5] showed that, under the strong no-overbidding assumption, the Bayesian PoA of the simultaneous Vickrey auction is strictly greater than 2, and furthermore the price of anarchy is $\Omega(n^{1/4})$ when agent values are allowed to be correlated. In the full version of the paper we show that similar results hold also under the weak no-overbidding assumption, proving bounds strictly greater than 2 and $\Omega(n^{1/6})$, respectively.

Our constant bounds hold for subadditive bidders, whereas constant bounds on Bayesian price of anarchy were previously known only for the subclass of fractionally subadditive (i.e. XOS) valuations [17]. Previous work that attempted to bound the BPoA for subadditive valuations [5, 34] provided constant bounds for XOS valuations, then used the logarithmic factor separation between XOS and subadditive valuations to establish a logarithmic upper bound on the BPoA for subadditive valuations. While it seems plausible to use the close relation between XOS and subadditive valuations, any analysis that follows this trajectory would encounter this inevitable logarithmic gap. The challenge, therefore, is in developing a new proof technique for subadditive valuations, which does not go through XOS valuations. This is the approach taken in this work.

It should be noted that subadditive valuations are more expressive than their XOS counterparts, and obtaining price of anarchy bounds for subadditive valuations is significantly more challenging. In particular, for XOS valuations, a player who aims to win a certain set S has a natural choice of bid: the additive valuation that determines his value for set S. For subadditive valuations, there is no such notion of a natural bid aimed at representing one's value for a particular set, and hence even determining how best to bid on a certain set of interest is a non-trivial task.

Related Work Combinatorial auctions is a canonical subject of study in algorithmic mechanism design (see 44 and references therein for the large body of literature on this subject). While most previous work focuses on the design of truthful mechanisms, we follow the more recent literature on the analysis of simple and practical (albeit not truthful) auctions.

Following the rich literature on the *price of anarchy* (PoA) [see, e.g., 35, 49, for references], Christodoulou et al. [17] pioneered the study of the *Bayesian price of anarchy* (BPoA) and applied it to item-bidding auctions. They bounded the BPoA by 2 in simultaneous second-price auctions with XOS valuations, which are equivalent to fractionally subadditive functions [24]. The same bound was extended to the more general class of subadditive valuations by Bhawalkar and Roughgarden [5], and later to general valuations by Fu et al. [27], albeit only with respect to *pure* equilibria (when they exist). The price of anarchy was studies also in simultaneous first-price auctions by Hassidim et al. [34], who showed a pure PoA of 1 for general valuations[4], and a constant BPoA for XOS valuations. The effect of the underlying single-item auction on the PoA was further studied by Bhawalkar and Roughgarden [6].

For both first- and second-price simultaneous auctions, the BPoA for subadditive valuations was not previously known to be better than $O(\log n)$. Previous techniques applied the constant bounds for XOS valuations, using the $O(\log n)$ separation between XOS and subadditive valuations [see e.g. 5].

Studies on PoA and BPoA have provided insights into other settings, e.g. auctions employing greedy algorithms [38], Generalized Second Price Auctions [37, 39, 14], uniform-price multi-unit auctions [41], and network formation settings [3].

The *smoothness* technique for Bayesian games, developed by Rahman [46] and Syrgkanis [51], provides a method for extending bounds on pure PoA to Bayesian PoA. However, to the best of our knowledge, our approach does not fall within ehis framework. Roughly speaking, the smoothness framework requires that each player can find a good "default" strategy given his type, which is independent of the opponents' strategy selections. However, subadditive valuations do not seem to admit such bids,[5] and indeed the strategies we consider in our analysis depend heavily on the distribution of strategies applied by all players at equilibrium.

[4]Pure Nash equilibria rarely exist in this case though, as they are shown to be equivalent to Walrasian equilibria of the corresponding two-sided market.

[5]We note that one can apply the technique on XOS valuations, but because of the $O(\log n)$ separation between XOS and subadditive valuations [see e.g. 5] this gives only a logarithmic bound.

Organization of the chapter We introduce the necessary background and notation in Section 5.2. Our analysis then proceeds in two parts. In the first part, Section 5.3, we consider a single-player game in which the player, a subadditive buyer, must determine how best to bid on a set of objects against a distribution over price vectors. We show that, for every distribution for which the expected sum of prices is not too large, the buyer has a bidding strategy that guarantees a high expected utility (compared to the player's value for the set of all objects).

In the second part of our analysis for the first-price (Section 5.4) and Vickrey (Section 5.5) auctions, we show that every Bayes-Nash equilibrium must have high expected social welfare. We do this by considering deviations in which an agent uses the bidding strategy from the single-player game described in Section 5.3, applied to some subset of the objects. This subset of objects is chosen randomly: agent i draws a new profile of types for his opponents from the type distribution, then considers bidding for the set he would be allocated under this "virtual" type profile. At a BNE, agent i cannot benefit from such a randomized deviation; we show that this implies that the social welfare at equilibrium is at least a constant times the optimal welfare.

5.2 Preliminaries

5.2.1 Combinatorial Auctions and Equilibria

Combinatorial Auctions In a combinatorial auction, m items are sold to n bidders. Each bidder has a private combinatorial valuation captured by a set function $v : 2^{[m]} \to \mathbb{R}$ over different bundles $S \subseteq [m]$. Throughout the paper we assume the valuations are *monotone*, i.e. for every subset $S \subseteq T \subseteq [m]$ it holds that $v(S) \leq v(T)$. In a *Bayesian* (partial-information) setting, the bidders' valuation profile $\boldsymbol{v}$ is drawn from a commonly known product distribution[6] $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_n$. The outcome of an auction consists of an allocation $\boldsymbol{x} = (x_1, \cdots, x_n) \in 2^{[m]\times n}$, where x_i is the bundle of items allocated to bidder i, and payments made by each bidder. The *social welfare* of an allocation is $\sum_{i\in[n]} v_i(x_i)$. For any given valuation profile $\boldsymbol{v}$, we let $(\mathrm{OPT}_1^{\boldsymbol{v}}, \ldots, \mathrm{OPT}_n^{\boldsymbol{v}})$ denote the welfare-maximizing assignment for profile $\boldsymbol{v}$.

Simultaneous Item-Bidding Auctions In a simultaneous item-bidding auction, each bidder simultaneously submits a vector of bids, one for each item. The outcome of the auction is then determined item by item according to the bids placed on each item. In this paper we study two forms of such auctions: *simultaneous first price auctions* and *simultaneous second price auctions*.[7] In both auctions, each item is allocated to the bidder who has placed the highest bid on it (breaking ties arbitrarily but consistently). In a (simultaneous) first price auction, the winner of each item pays his bid on that item, while in a (simultaneous) second price auction, the winner of each item pays the second highest bid on that item. We now give a more formal description of this process.

We generally write $b_i(j)$ to denote the bid of player i on item j, and $\vec{b}_i$ for the vector of bids placed by bidder i. Alternatively, we may think of agent i's bid b_i as an additive function $b_i(S) = \sum_{j\in S} b_i(j)$ that corresponds[8] to the bid-vector $\vec{b}_i$. Given a sequence of bid profiles $\mathbf{b} = (b_1, \ldots, b_n)$, we write $W_i(\mathbf{b})$ for the set of items won by bidder i, and $\vec{p}_i \in \mathbb{R}^m$ the vector of payments made by bidder i on the items. In this notation, the first-

[6]Whenever an expectation is taken with respect to valuations, it will be assumed that they are drawn from these corresponding distributions.

[7]The word "simultaneous" is often omitted, as we study only simultaneous (in contrast to sequential) auctions.

[8]There is an easy equivalence between an additive function $a(S) := \sum_{j\in S} a(\{j\})$ and its concise vector description $\vec{a} = (a(\{1\}), \ldots, a(\{m\}))$. We will use functional and vector representations interchangeably as the situation demands.

and second-price auctions can be summarized as follows:

First-price	Vickrey
won set: $W_i(\mathbf{b}) = \{j \in [m] \mid b_i(j) > b_k(j), \forall k \neq i\}$	
payment: $p_i(j) =$	
$\begin{cases} b_i(j), & j \in W_i(\mathbf{b}) \\ 0, & j \notin W_i(\mathbf{b}) \end{cases}$	$\begin{cases} \max\limits_{k \neq i} b_k(j), & j \in W_i(\mathbf{b}) \\ 0, & j \notin W_i(\mathbf{b}) \end{cases}$

We assume bidders have quasi-linear utilities, i.e. the *utility* of bidder i for a given bid profile $\mathbf{b}$ is given by $u_i(\mathbf{b}) = v_i(W_i(\mathbf{b})) - p_i(W_i(\mathbf{b}))$.

A Single Bidder's Perspective on Bidding In both first- and second-price auctions, the set of items won by a bidder i bidding b_i is determined solely by a coordinate-wise comparison between b_i and the largest bid placed by the other bidders. Let $\varphi_i(\mathbf{b}_{\text{-}i})$ be the vector whose j-th component is $\max_{k \neq i} b_k(j)$. It is often convenient to write $W(b_i, \mathbf{b}_{\text{-}i})$ as $W(b_i, \vec{p})$ where $\vec{p} = \varphi_i(\mathbf{b}_{\text{-}i})$. We think of $\vec{p}$ as the vector of prices perceived by bidder i: in the second price auction, the bidder pays the price on an item if his bid exceeds it; and in the first price auction the bidder pays his own bid on such an item, and $\vec{p}$ is the minimum such winning bid. It is in this light that we often write $\varphi_i(\mathbf{b}_{\text{-}i})$ as prices $\vec{p}$ when this causes no confusion. We will also shorten the notation $v(W(b, \vec{p}))$ to $v(b, \vec{p})$, meaning the value obtained when bidding b against perceived prices $\vec{p}$.

Strategies and Equilibria Buyers select their bids strategically in order to maximize utility. The bidding behavior of a buyer given its valuation is described by a *strategy*. A strategy s_i maps each valuation v_i to a distribution over bid vectors; we interpret $s_i(v_i)$ as the (possibly randomized) set of bids placed by bidder i when his type is v_i.

Definition 5.2.1. (**Bayes-Nash Equilibrium**) A profile of strategies $\mathbf{s} = (s_1(v_1), \ldots, s_n(v_n))$ is in *Bayes-Nash equilibrium* (BNE) for distribution $\mathcal{F}$ if, for every buyer i, type v_i, and bidding strategy $\widetilde{s_i}$,

$$\mathbf{E}_{\mathbf{v}_{-i}}\left[\mathop{\mathbf{E}}_{\substack{\mathbf{b}_{-i}\sim\mathbf{s}(\mathbf{v}_{-i}),\\ b_i\sim s_i(v_i)}}[u_i(b_i, \mathbf{b}_{-i})]\right] \geq \mathbf{E}_{\mathbf{v}_{-i}}\left[\mathop{\mathbf{E}}_{\substack{\mathbf{b}_{-i}\sim\mathbf{s}(\mathbf{v}_{-i}),\\ \widetilde{b_i}\sim\widetilde{s_i}}}\left[u_i(\widetilde{b_i}, \mathbf{b}_{-i})\right]\right].$$

Given Fubini's Theorem, we can shorten the condition as follows (such shorthand forms are used throughout the paper):

$$\mathbf{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[u_i(\mathbf{b})] \geq \mathbf{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v}),\widetilde{b_i}\sim\widetilde{s_i}}\left[u_i(\widetilde{b_i}, \mathbf{b}_{-i})\right]. \tag{5.1}$$

Definition 5.2.2. (**Bayesian Price of Anarchy**) Given an auction type (either first- or second-price), the *Bayesian price of anarchy* (BPoA) is the worst-case ratio between the expected optimal welfare and the expected welfare at a BNE and is given by

$$\max_{\substack{(\mathcal{F},\ \mathbf{s}):\\ \mathbf{s}\text{ a BNE for }\mathcal{F}}} \frac{\mathbf{E}_{\mathbf{v}}[\sum_i v_i(\mathrm{OPT}_i^{\mathbf{v}})]}{\mathbf{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[\sum_i v_i(W_i(\mathbf{b}))]}.$$

For second price auctions we will consider BPoA under natural restrictions on the strategies used by the bidders. In such cases, the maximum in Definition 5.2.2 is taken with respect to BNE under that restricted class of strategies. We note that a BNE is guaranteed to exist as long as the space of valuations and potential bids is discretized, say with all values expressed as increments of some $\epsilon > 0$. A more detailed discussion of BNE existence appears in the full version of the paper.

5.2.2 Subadditive Valuations

We focus on valuations that are complement-free in the following general sense:

Definition 5.2.3. A set function $v : 2^{[m]} \to \mathbb{R}_+$ is *subadditive* if, for any subsets $S_1, S_2 \subset [m]$,

$$v(S_1) + v(S_2) \geq v(S_1 \cup S_2).$$

The class of subadditive functions strictly includes a hierarchy of more restrictive complement-free functions such as submodular and gross substitute functions (see 36 for definitions and discussions). Among these, the XOS functions, as defined below, have a particular kinship with subadditive functions. XOS literally means XOR (taking the maximum) of OR's (taking sums), and this class of valuations is known to be equivalent to the class of *fractionally subadditive* functions [24].

Definition 5.2.4. A function $v : 2^{[m]} \to \mathbb{R}_+$ is said to be *XOS* if there exists a collection of additive functions $a_1(\cdot), \ldots, a_k(\cdot)$ (that is, $a_i(S) \coloneqq \sum_{j \in S} a_i(\{j\})$ for every set $S \subset [m]$), such that for each $S \subseteq [m]$, $v(S) \coloneqq \max_{1 \leq i \leq k} a_i(S)$.

One of the characterizations of XOS functions uses the following definition.

Definition 5.2.5. A function $f(\cdot)$ is said to be *dominated* by a set function $g(\cdot)$ if for any subset $S \subseteq [m]$, $f(S) \leq g(S)$. We say that a vector $\vec{a} = (a_1, \ldots, a_m)$ is dominated by a set function $v(\cdot)$, if as an additive function $a(\cdot)$ is dominated by $v(\cdot)$.

It is not too difficult to observe that $v(\cdot)$ is XOS if and only if for every set $T \subset [m]$ there is an additive function $a(\cdot)$ dominated by $v(\cdot)$ such that $a(T) = v(T)$.

For a general subadditive function $v(\cdot)$, it can be the case that any additive function $a(\cdot)$ dominated by $v(\cdot)$ has $\Omega(\log(m))$ gap from $v([m])$, i.e. $\Omega(\log(m))a([m]) \leq v([m])$, (See 5 for such an example) and a logarithmic factor is also an upper bound. Previous work that attempted to bound the BPoA for subadditive valuations [5, 34] provided constant bounds for XOS valuations, then used the logarithmic factor separation between XOS and subadditive valuations to establish a logarithmic upper bound on the BPoA for subadditive valuations. In order to establish a constant bound for subadditive valuations, we turn to a different technique.

5.2.3 Overbidding

It is well known that in second price auctions, even with only a single item, the price of anarchy can be infinite when bidders are not restricted in their bids.[9] To exclude such pathological cases, previous literature [e.g. 17, 5] has made the following *no-overbidding* assumption standard:[10]

Definition 5.2.6. A bidder is *strongly no-overbidding* if his bid $b(\cdot)$ is dominated by his valuation $v(\cdot)$.

In other words, a bidder is guaranteed to derive non-negative utility, no matter what are the prices in the market. Thus strong no overbidding is a strong risk-aversion assumption on the buyers. One may also consider less risk concerned bidders—in the following we generalize a weaker assumption of no-overbidding introduced by Fu et al. [27].

Definition 5.2.7. Given a price distribution F, a bidder is said to be *weakly no-overbidding* if his bid vector b satisfies
$\mathbf{E}_{p\sim F}[v(W(b,p))] \geq \mathbf{E}_{p\sim F}[b(W(b,p))]$, where $W(b,p)$ denotes the subset of items he wins when he bids b at price p, i.e., $W(b,p) = \{j \in [m] \mid b(j) \geq p(j)\}$.

We will bound BPoA under both weakly and strongly no-overbidding assumptions for simultaneous second price auctions (note that these sets are incomparable).

5.3 Bidding Under Uncertain Prices

As discussed in Section 5.2, a bidder in a simultaneous auction faces the problem of maximizing his utility in presence of uncertain prices (which are the largest bids placed

[9] A canonical example is two bidders who value the item at 0 and a large number h, respectively, but the first bidder bids $h+1$ and the second bidder bids 0.

[10] We note that such no-overbidding assumptions were also made in other contexts [e.g. 38, 37].

by other bidders). While this maximization problem is intricate, we show in this section particular bidding strategies that result in utilities comparable with the bidder's value of the whole bundle minus the expected total prices. In other words, given a price distribution F, it is desired to have a bidding strategy b such that

$$\mathbf{E}_{\mathbf{p}\sim F}\left[v(b,p)\right] - b([m]) \geq \alpha v([m]) - \mathbf{E}_{p\sim F}\left[p([m])\right], \tag{5.2}$$

for some constant α. Such bidding strategies are key ingredients of the BPoA proofs in later sections, and may also be of independent interest.

For fixed prices, achieving (5.2) is trivial, even for $\alpha = 1$; indeed, given a price vector $\vec{p}$, by bidding according to $b = p$, a bidder obtains $v(b,p) - b([m]) = v([m]) - p([m])$. The case in which prices are drawn at random is more intricate, and is the subject of the remainder of this section.

Lemma 5.3.1 (**Bidding against a price distribution**). *For any distribution F of prices p and any subadditive valuation $v(\cdot)$ there exists a bid b_0 such that*

$$\mathbf{E}_{p\sim F}\left[v(b_0,p)\right] - b_0([m]) \geq \frac{1}{2}v([m]) - \mathbf{E}_{p\sim F}\left[p([m])\right]. \tag{5.3}$$

Proof. We show a random bidding strategy that guarantees the desired inequality in expectation, and infer the existence of a bid, drawn from the suggested distribution, that achieves the same inequality. Consider a bid that is drawn according to the exact same distribution as the prices. It holds that

$$\begin{aligned}
\mathbf{E}_{b\sim F}\left[\mathbf{E}_{p\sim F}\left[v(b,p)\right]\right] &= \mathbf{E}_{p\sim F}\left[\mathbf{E}_{b\sim F}\left[v(b,p)\right]\right] \\
&= \frac{1}{2}\,\mathbf{E}_{b\sim F}\left[\mathbf{E}_{p\sim F}\left[v(b,p) + v(p,b)\right]\right] \\
&\geq \frac{1}{2}\,\mathbf{E}_{b\sim F}\left[\mathbf{E}_{p\sim F}\left[v([m])\right]\right] \\
&= \frac{1}{2}v([m]),
\end{aligned}$$

where the inequality follows from subadditivity (which guarantees that $v(b,p)+v(p,b) \geq$

$v([m])$ for every p and b). Using the last inequality, it follows that

$$\begin{aligned}\mathbf{E}_{b\sim F}\left[\mathbf{E}_{p\sim F}\left[v(b,p)\right]-b([m])\right] &\geq \frac{1}{2}v([m])-\mathbf{E}_{b\sim F}\left[b([m])\right]\\ &= \frac{1}{2}v([m])-\mathbf{E}_{p\sim F}\left[p([m])\right].\end{aligned}$$

Since a bid drawn from $\mathbf{F}$ satisfies (5.3) in expectation, there must exist a bid b_0 satisfying (5.3). □

Safe Bidding Under Uncertainty

As noted in Section 5.2.3, in order to obtain any meaningful bound on BPoA for second price auctions, one needs to assume that bidders are not overbidding. Unfortunately, Lemma 5.3.1 is not concerned with such requirements. This problem is addressed in Lemma 5.3.3, where it is shown that a strongly no-overbidding strategy analogous to that in Lemma 5.3.1 always exists.

Notably, when the no-overbidding requirement is imposed, the existence of a bid satisfying (5.2) is nontrivial even for the case in which the prices are fixed. The following lemma, rephrased from 5, establishes its existence:

Lemma 5.3.2 (**follows from Lemma 3.3 in 5**). *For a given price vector p and any subadditive valuation $v(\cdot)$ there exists a bid b dominated by $v(\cdot)$ such that*

$$v(b,p)-b([m])\geq v([m])-p([m]).$$

We now turn to analyze the case of random prices.

Lemma 5.3.3 (**No Overbidding Against Price Distributions**). *For any distribution F of prices p and any subadditive valuation $v(\cdot)$ there exists a bid b_0 dominated by $v(\cdot)$ such that*

$$\mathbf{E}_{p\sim F}\left[v(b_0,p)\right]-b_0([m])\geq \frac{1}{2}v([m])-\mathbf{E}_{p\sim F}\left[p([m])\right]. \tag{5.4}$$

Proof. Let q be any price vector in the support of the distribution F. Let $T \subseteq [m]$ be a maximal set such that $v(T) \leq q(T)$. We consider a *truncated* price vector $\widetilde{q}$, which is set to 0 on the coordinates corresponding to T, and coincides with q on the coordinates corresponding to $[m] \setminus T$.

We first observe that $\widetilde{q}$ **is dominated by** $v(\cdot)$. Indeed, for any set $R \subset [m] \setminus T$ it holds that $v(R) > q(R)$, since otherwise

$$v(R \cup T) \leq v(R) + v(T) \leq q(R) + q(T) = q(R \cup T),$$

in contradiction to the fact that T is a maximal set satisfying $v(T) \leq q(T)$.

We next establish that for any bid b, it holds that

$$v(b, q) + q([m]) \geq v(b, \widetilde{q}) + \widetilde{q}([m]). \tag{5.5}$$

Indeed, we have $W(b, \widetilde{q}) \subseteq W(b, q) \cup T$. Therefore, $v(b, \widetilde{q}) \leq v(b, q) + v(T)$ due to subadditivity of $v(\cdot)$. Now (5.5) follows by observing that $q([m]) - \widetilde{q}([m]) = q(T) \geq v(T)$.

We next define the distribution $\widetilde{\mathcal{D}} \coloneqq \{\widetilde{q} \mid q \sim F\}$, which consists of truncated prices drawn from F. Equation (5.5) now extends for any bid b to

$$\mathbf{E}_{p \sim F}\left[v(b, p) + p([m])\right] \geq \mathbf{E}_{\widetilde{p} \sim \widetilde{\mathcal{D}}}\left[v(b, \widetilde{p}) + \widetilde{p}([m])\right]. \tag{5.6}$$

Recall that each $\widetilde{q} \sim \widetilde{\mathcal{D}}$ is dominated by $v(\cdot)$, therefore, bidding any b drawn from $\widetilde{\mathcal{D}}$ satisfies the strongly no overbidding requirement. Furthermore, by applying (5.6) to each $b \sim \widetilde{\mathcal{D}}$ we get

$$\begin{aligned}
\mathop{\mathbf{E}}_{b \sim \widetilde{\mathcal{D}}}\Big[\mathbf{E}_{p \sim F}\left[v(b, p) + p([m])\right]\Big] &\geq \mathop{\mathbf{E}}_{b \sim \widetilde{\mathcal{D}}}\Big[\mathbf{E}_{\widetilde{p} \sim \widetilde{\mathcal{D}}}\left[v(b, \widetilde{p}) + \widetilde{p}([m])\right]\Big] \\
&= \mathbf{E}_{b \sim \widetilde{\mathcal{D}}}\Big[\mathbf{E}_{\widetilde{p} \sim \widetilde{\mathcal{D}}}\left[v(b, \widetilde{p})\right]\Big] + \mathbf{E}_{b \sim \widetilde{\mathcal{D}}}\left[b([m])\right] \\
&\geq \frac{1}{2} v([m]) + \mathbf{E}_{b \sim \widetilde{\mathcal{D}}}\left[b([m])\right],
\end{aligned}$$

where the last inequality follows in a manner similar to the proof of Lemma 5.3.1. The assertion of the lemma follows. □

5.4 BPoA of First Price Auctions

In this section we apply the bidding strategy from Lemma 5.3.1 to bound the Bayesian price of anarchy of simultaneous first-price auctions.

Theorem 5.4.1. *In a simultaneous first-price auction with subadditive bidders, the Bayesian price of anarchy is at most* 2*.*

Proof. We begin with a brief outline of the proof. Our plan is to fix a Bayes-Nash equilibrium and then consider, for each agent, a potential deviating strategy. This deviation will use the bidding strategy from Lemma 5.3.1, applied to some subset of the objects. To determine which subset to bid upon, each agent i will do the following: given her own value v_i, she will draw a "virtual" type profile $\boldsymbol{v}^*_{-i}$ for the other agents from distribution $\mathcal{F}$, and then bid upon the set that she would be assigned in the optimal allocation for $(v_i, \boldsymbol{v}^*_{-i})$. To determine how to bid upon this set, she draws a second type profile for the other agents, $\boldsymbol{v}_{-i}$, as dictated by Lemma 5.3.1. At BNE, agent i cannot benefit from such a randomized deviation; this implies a bound on the expected utility of each agent at equilibrium (inequality (5.8)). By taking a sum over all agents and using linearity of expectation to disentangle the random variables $\boldsymbol{v}$ and $\boldsymbol{v}'^*$, we show that this implies the social welfare at equilibrium is at least a constant times the optimal welfare.

We now proceed with the details, beginning with notation. Fix type distributions $\mathcal{F} = \prod_{i=1}^n \mathcal{F}_i$ and let $\mathbf{s}$ be a BNE for $\mathcal{F}$. Fix an agent i and an arbitrary subadditive valuation v_i. Fix an arbitrary $\boldsymbol{v}_{-i}$, and let $\boldsymbol{v} = (v_i, \boldsymbol{v}_{-i})$. Fix an arbitrary $\boldsymbol{v}^*_{-i}$, and let $\boldsymbol{v}^* = (v_i, \boldsymbol{v}^*_{-i})$. Recall that $(\mathrm{OPT}_1^{\boldsymbol{v}^*}, \dots, \mathrm{OPT}_n^{\boldsymbol{v}^*})$ is the welfare-optimal allocation for $\boldsymbol{v}^*$.

Recall that each bid profile $\mathbf{b}_{\text{-}i}$ induces a price vector $\varphi_i(\mathbf{b}_{\text{-}i})$ on bidder i. Let $\vec{p}$ be equal to $\varphi_i(\mathbf{b}_{\text{-}i})$ on $\mathrm{OPT}_i^{\boldsymbol{v}^*}$ and 0 elsewhere. Let $\mathbf{F}$ be the distribution over these price vectors $\vec{p} = \vec{p}(\mathbf{b}_{\text{-}i})$, where $\mathbf{b} \sim \mathbf{s}(\boldsymbol{v})$. That is, $\mathbf{F}$ is precisely the distribution over the maximum bids on the items in $\mathrm{OPT}_i^{\boldsymbol{v}^*}$, excluding the bid of player i. Note that $\boldsymbol{v}^*$, which is different from $\boldsymbol{v}$, was used only to determine the set $\mathrm{OPT}_i^{\boldsymbol{v}^*}$, whereas $\boldsymbol{v}$ determines the distribution of prices over the items in $\mathrm{OPT}_i^{\boldsymbol{v}^*}$. Much of the following proof involves handling and, to some extent, disentangling the two. By replacing $[m]$ by $\mathrm{OPT}_i^{\boldsymbol{v}^*}$ in Lemma 5.3.1, there exists a bid vector b_i' over the objects in $\mathrm{OPT}_i^{\boldsymbol{v}^*}$ such that, thinking now of p as an additive function,

$$\begin{aligned}
&\mathbf{E}_{p\sim F}\left[v_i(b_i', p)\right] - b_i'(\mathrm{OPT}_i^{\boldsymbol{v}^*}) \\
&\geq \frac{1}{2} v_i(\mathrm{OPT}_i^{\boldsymbol{v}^*}) - \mathbf{E}_{p\sim F}\left[p(\mathrm{OPT}_i^{\boldsymbol{v}^*})\right].
\end{aligned} \tag{5.7}$$

Since $\mathbf{s}$ forms a BNE, we have that

$$\begin{aligned}
\mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[u_i(\mathbf{b})\right] \geq \mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[u_i(b_i', \mathbf{b}_{\text{-}i})\right]& \\
= \mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[v_i(b_i', \varphi_i(\mathbf{b}_{\text{-}i}))\right] - \mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[b_i'(W_i(b_i', \mathbf{b}_{\text{-}i}))\right]& \\
\geq \mathbf{E}_{p\sim F}\left[v_i(b_i', p)\right] - b_i'(\mathrm{OPT}_i^{\boldsymbol{v}^*}),&
\end{aligned}$$

where the last inequality follows from the definition of $\mathbf{F}$ and the fact that $W_i(b_i', \mathbf{b}_{\text{-}i}) \subseteq \mathrm{OPT}_i^{\boldsymbol{v}^*}$ for all $\mathbf{b}_{\text{-}i}$. Applying (5.7) and the definition of $p \sim \mathbf{F}$, we conclude that

$$\begin{aligned}
&\mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[u_i(\mathbf{b})\right] \\
&\geq \frac{1}{2} v_i(\mathrm{OPT}_i^{\boldsymbol{v}^*}) - \mathop{\mathbf{E}}_{\substack{\boldsymbol{v}_{-i},\\ \mathbf{b}_{\text{-}i}\sim\mathbf{s}_{-i}(\boldsymbol{v}_{-i})}}\left[\sum_{j\in\mathrm{OPT}_i^{\boldsymbol{v}^*}} \max_{k\neq i} b_k(j)\right].
\end{aligned} \tag{5.8}$$

Taking the sum over all i and expectations over all $v_i \sim \mathcal{F}_i$ and $\boldsymbol{v}_{-i}^* \sim \mathcal{F}_{-i}$, we conclude that

$$\begin{aligned}
\sum_i \mathop{\mathbf{E}}_{\substack{\boldsymbol{v},\boldsymbol{v}_{-i}^*,\\ \mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}}\left[u_i(\mathbf{b})\right] \geq \frac{1}{2}\sum_i \mathop{\mathbf{E}}_{v_i,\boldsymbol{v}_{-i}^*}\left[v_i(\mathrm{OPT}_i^{\boldsymbol{v}^*})\right]& \\
- \sum_i \mathop{\mathbf{E}}_{\substack{\boldsymbol{v},\boldsymbol{v}_{-i}^*,\\ \mathbf{b}_{\text{-}i}\sim\mathbf{s}_{-i}(\boldsymbol{v}_{-i})}}\left[\sum_{j\in\mathrm{OPT}_i^{\boldsymbol{v}^*}} \max_{k\neq i} b_k(j)\right]&.
\end{aligned} \tag{5.9}$$

Let us consider each of the three terms of (5.9) in turn. The LHS is equal to $\mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}[\sum_i u_i(\mathbf{b})]$, as $\boldsymbol{v}_{-i}^*$ does not appear inside the expectation. The first term on the

RHS is equal to $\frac{1}{2}\,\mathbf{E}_{\boldsymbol{v}}[\sum_i v_i(\mathrm{OPT}_i^{\boldsymbol{v}})]$, by relabeling $\boldsymbol{v}^*_{-i}$ by $\boldsymbol{v}_{-i}$. For the final term on the RHS of (5.9), we note that

$$\begin{aligned}
\sum_i \mathop{\mathbf{E}}_{\substack{\boldsymbol{v},\boldsymbol{v}^*_{-i},\\ \mathbf{b}_{-i}\sim\mathbf{s}_{-i}(\boldsymbol{v}_{-i})}} &\left[\sum_{j\in\mathrm{OPT}_i^{\boldsymbol{v}^*}} \max_{k\neq i} b_k(j)\right] \\
&\leq \sum_i \mathop{\mathbf{E}}_{\substack{\boldsymbol{v},\boldsymbol{v}^*_{-i},\widehat{v}_i,\\ \mathbf{b}\sim\mathbf{s}(\widehat{v}_i,\boldsymbol{v}_{-i})}} \left[\sum_{j\in\mathrm{OPT}_i^{\boldsymbol{v}^*}} \max_{k} b_k(j)\right] \\
&= \mathop{\mathbf{E}}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_j \max_k b_k(j)\right],
\end{aligned}$$

where the first inequality follows due to the fact we take a maximum over a larger set, and the last equality follows from the fact that $\mathrm{OPT}_i^{\boldsymbol{v}^*}$ imposes a partition over $[m]$, and by relabeling. We note a subtlety: in the first line we select a bid vector $\mathbf{b}$ with respect to $(\widehat{v}_i, \boldsymbol{v}_{-i})$, rather than $(v_i, \boldsymbol{v}_{-i})$, so that $\mathbf{b}$ is independent of the partition $(\mathrm{OPT}_1^{\boldsymbol{v}^*}, \ldots, \mathrm{OPT}_n^{\boldsymbol{v}^*})$. Applying these simplifications to the terms of (5.9), we conclude that

$$\begin{aligned}
\mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_i u_i(\mathbf{b})\right] \geq \frac{1}{2}\,\mathbf{E}_{\boldsymbol{v}}&\left[\sum_i v_i(\mathrm{OPT}_i^{\boldsymbol{v}})\right] \\
&- \mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_j \max_k b_k(j)\right].
\end{aligned} \tag{5.10}$$

Since we are in a first-price auction setting, it holds that

$$\begin{aligned}
\mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_i u_i(\mathbf{b})\right] = \mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}&\left[\sum_i v_i(W_i(\mathbf{b}))\right] \\
&- \mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_j \max_k b_k(j)\right].
\end{aligned}$$

Equation (5.10) therefore implies that

$$\mathbf{E}_{\boldsymbol{v},\mathbf{b}\sim\mathbf{s}(\boldsymbol{v})}\left[\sum_i v_i(W_i(\mathbf{b}))\right] \geq \frac{1}{2}\,\mathbf{E}_{\boldsymbol{v}}\left[\sum_i v_i(\mathrm{OPT}_i^{\boldsymbol{v}})\right],$$

which yields the desired result. □

Remark: In the full version of the paper we show that the upper bound does not carry over to the case where the bidders' valuations are correlated. Specifically, a polynomial lower bound of $\Omega(n^{1/6})$ is given on the Bayesian price of anarchy for this case. The construction is based on a lower bound due to Bhawalkar and Roughgarden [5] for second-price auctions.

5.5 BPoA of Second Price Auctions

We now turn to the case of simultaneous second-price auctions. We show that the Bayesian price of anarchy of such an auction is always at most 4 for subadditive bidders, assuming that bidders' valuations are independent and bidders select strategies that satisfy either the strong or weak no-overbidding assumption.

Theorem 5.5.1. *In simultaneous second-price auctions where bidders have subadditive valuations, and every bidder is either strongly or weakly no-overbidding, the Bayesian price of anarchy is at most* 4.

Proof. Fix type distributions $\mathcal{F}$ and let $\mathbf{s}$ be a BNE for $\mathcal{F}$. We can then derive inequality (5.10) in precisely the same way as in the proof of Theorem 5.4.1 (using now Lemma 5.3.3 instead of Lemma 5.3.1); we then have that

$$\begin{aligned}\mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}\left[\sum_i u_i(\mathbf{b})\right] \geq \frac{1}{2}\,\mathbf{E}_v\left[\sum_i v_i(\mathrm{OPT}_i^v)\right] \\ - \mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}\left[\sum_j \max_k b_k(j)\right].\end{aligned} \tag{5.11}$$

Note that $\mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}[\sum_i v_i(W_i(\mathbf{b}))] \geq \mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}[\sum_i u_i(\mathbf{b})]$. Also, since each agent i is assumed to be strongly or weakly no overbidding, it holds that

$$\begin{aligned}\mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}\left[\sum_j \max_k b_k(j)\right] &= \mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}\left[\sum_i \sum_{j\in W_i(\mathbf{b})} b_i(j)\right] \\ &\leq \mathbf{E}_{v,\mathbf{b}\sim\mathbf{s}(v)}\left[\sum_i v_i(W_i(\mathbf{b}))\right].\end{aligned}$$

Equation (5.11) therefore implies that

$$\mathbf{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_i v_i(W_i(\mathbf{b}))\right] \geq \frac{1}{2}\,\mathbf{E}_{\mathbf{v}}\left[\sum_i v_i(\mathrm{OPT}_i^{\mathbf{v}})\right] - \mathbf{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_i v_i(W_i(\mathbf{b}))\right],$$

as required. □

Bhawalkar and Roughgarden [5] showed that the Bayesian price of anarchy of second price auctions can be strictly worse than the pure price of anarchy when bidders are strongly no overbidding. In what follows we give an example showing that such a gap exists also when bidders are weakly no overbidding. We note that this gap is not implied by the example given by Bhawalkar and Roughgarden since the strategy profile in their example is not a BNE under the weaker no overbidding notion (as can be easily verified). The full analysis of the example appears in the full version of the paper; the following is a sketch.

Example 5.5.1 (Bayesian price of anarchy can be strictly larger than 2 when bidders are weakly no overbidding and have subadditive valuations). Consider an instance with 2 bidders and 6 items, where the set of items is divided into two sets, of 3 items each, denoted S_1 and S_2. Throughout, we shall present the example with parameters a and b for ease of presentation. The lower bound is obtained by substituting $a = 0.06$ and $b = 0.85$. In what follows, we describe the valuation function of bidder 1; bidder 2's valuation is symmetric w.r.t. the sets S_1 and S_2. Bidder 1's valuation over the items in S_1 is additive with respective values (over the 3 items) of $(a, a, b), (b, a, a)$ or (a, b, a), each with probability $1/3$. Bidder 1's valuation over the items in S_2 is 2 if she gets all three items, and 1 for any non-empty strict subset of S_2. Bidder 1's valuation for an arbitrary subset T the maximum of her value for $T \cap S_1$ and her value for $T \cap S_2$. One can verify that this is indeed a subadditive valuation function.

We claim that the profile in which each bidder i bids her true (additive) valuation on S_i and 0 on all other items is a Bayes-Nash equilibrium with weakly no overbidding bidders for the specified parameter values. The full proof is deferred to the full version of the paper, where it is shown that the only beneficial deviations break the weakly no-overbidding assumption. Under this bidding profile, each bidder derives a utility of $2a + b$, amounting to a social welfare of $2(2a + b) = 1.94$. In contrast, if bidder 1 is allocated S_2 and bidder 2 is allocated S_1, then each bidder derives a utility of 2, amounting to a social welfare of 4. Consequently, the Bayesian price of anarchy is $4/1.94 > 2.061$.

APPENDIX A

PROOFS FROM Chapter 4

A.1 Single-dimensional Proofs

We prove that single-dimensional linear agents are revenue linear. In the proof below $P(\hat{q})$ denotes the expected revenue from posting price $V(\hat{q})$, i.e., $P(\hat{q}) = \hat{q} \cdot V(\hat{q})$.

Proof of Theorem 4.2.2. Given the setup in Section 4.2 it suffices to upper bound the optimal revenue by the marginal revenue. Consider the following sequence of steps, the second of which invokes revenue equivalence. Suppose we optimize for $\hat{x}$ and get some (possibly more restrictive) allocation rule x, then x better be a fixed point of $\text{Rev}[\cdot]$; moreover,

$$\text{Rev}[\hat{x}] = \text{Rev}[x].$$

By revenue equivalence, the revenue of any allocation rule is given by its price-posting revenue curve $P(\cdot)$, i.e.,

$$\text{Rev}[x] = \mathbf{E}_q\left[-x'(q) \cdot P(q)\right].$$

But, by definition $P(q) \leq R(q)$ for all q, so this revenue is upper bounded as,

$$\mathbf{E}_q\left[-x'(q) \cdot P(q)\right] \leq \mathbf{E}\left[-x'(q) \cdot R(q)\right] = \mathbf{E}\left[R'(q) \cdot x(q)\right],$$

which is equal to the marginal revenue for x. Applying integration by parts,[1]

$$\mathbf{E}\left[R'(q) \cdot x(q)\right] = \mathbf{E}\left[-R''(q) \cdot X(q)\right],$$

Because $X(q) \leq \hat{X}(q)$ for all q and "$-R''(q)$" is non-negative by the concavity of $R(q)$, we have,

$$\mathbf{E}\left[-R''(q) \cdot X(q)\right] \leq \mathbf{E}\left[-R''(q) \cdot \hat{X}(q)\right].$$

Integrating by parts (reverse of the above), we have the marginal revenue of $\hat{x}$.

$$\mathbf{E}\left[-R''(q) \cdot \hat{X}(q)\right] = \mathrm{MR}[\hat{x}]. \qquad \square$$

We conclude that the revenues are equal as, of course, $\mathrm{MR}[\hat{x}] \leq \mathrm{Rev}[\hat{x}]$.

A.2 Proofs from Section 4.3

In this appendix we prove Theorem 4.3.1.

Lemma (Lemma 4.3.2)**.** *For a linear single-agent problem, let x be the optimal allocation rule subject to some constraint $\hat{x}$. Then, for any $\hat{q}$ such that $R''(\hat{q}) \neq 0$ we have $X(\hat{q}) = \hat{X}(\hat{q})$.*

Proof. Since x is the optimal allocation rule subject to $\hat{x}$, we have $\mathrm{Rev}(x) = \mathrm{Rev}(\hat{x})$. Linearity implies that

$$\mathrm{MR}(\hat{x}) = \int_0^1 x(q)R'(q)\,dq = \int_0^1 \hat{x}(q)R'(q)\,dq = \mathrm{MR}(x).$$

Integrating by parts, we have

$$\left[X(q)R'(q)\right]_0^1 - \int_0^1 X(q)R''(q)\,dq = \left[\hat{X}(q)R'(q)\right]_0^1 - \int_0^1 \hat{X}(q)R''(q)\,dq. \tag{A.1}$$

Since $\hat{x}$ and x have the same ex ante probability of allocation $\hat{X}(1) = X(1)$; by definition $X(0) = \hat{X}(0) = 0$. By combining these observations with (A.1) we have

$$\int_0^1 X(q)R''(q)\,dq = \int_0^1 \hat{X}(q)R''(q)\,dq,$$

and therefore,

$$\int_0^1 [X(q) - \hat{X}(q)]R''(q)\,dq = 0. \tag{A.2}$$

Notice that for any q, $X(q) - \hat{X}(q)$ and $R''(q)$ are non-positive (by domination and concavity, respectively) so their product is non-negative. Therefore, (A.2) can be satisfied only if $[X(q) - \hat{X}(q)]R''(q) = 0$ for all q. This implies that if $R''(q) < 0$, then we must have $X(q) = \hat{X}(q)$, which completes the proof. $\square$

Lemma 4.3.2 in particular implies that for q with $R''(q) \neq 0$ the q-step mechanism (for step constraint $\hat{x}^{\hat{q}}$) has allocation rule $x^{\hat{q}} = \hat{x}^{\hat{q}}$. I.e., the q-step mechanism has full lotteries only (no partial lotteries).

For any such q, define T_q to be the set of types allocated (with full lotteries) in the optimal allocation subject to $\hat{x}^{\hat{q}}$. The following lemma shows that these sets are nested.

Lemma A.2.1. *For a revenue-linear single-agent problem, for any $q_1 > q_2$ and $R''(q_1), R''(q_2) \neq 0$, we must have $T_{q_1} \supseteq T_{q_2}$.*

Proof. Assume for contradiction that $T_{q_2} \setminus T_{q_1} \neq \emptyset$. Let $\alpha = F(T_{q_2} \setminus T_{q_1}) > 0$. Consider the following allocation constraint

$$\hat{x}(q) = \begin{cases} 1 & q \leq q_2 \\ 1/2 & q_1 \leq q \leq q_2 \\ 0 & q_1 \leq q. \end{cases}$$

By revenue linearity, the revenue of the optimal auction subject to $\hat{x}$ is $[R(q_1)+R(q_2)]/2$. Notice that the mechanism that runs $R(q_1)$ and $R(q_2)$ each with probability 1/2 achieves this revenue. The allocation rule x of this mechanism is

$$x(q) = \begin{cases} 1 & q \leq q_2 - \alpha \\ 1/2 & q_2 - \alpha \leq q \leq q_1 + \alpha \\ 0 & q_1 + \alpha \leq q. \end{cases}$$

Notice that this allocation rule is dominated by $\hat{x}$, and achieves the optimal revenue. Yet, we have

$$\hat{X}(q_1) = \int_{q=0}^{q_1} \hat{x}(q)dq > \int_{q=0}^{q_1} x(q)dq = X(q_1).$$

This contradicts Lemma 4.3.2. $\square$

A.3 Linearity

In this section we prove the linearity of the unit demand problem when the distribution of items is $U[0,1]^m$ for any number of items m. We use the following lemma first noted by Rochet [47].

Lemma A.3.1. *Utility function u corresponds to a truthful mechanism if and only if it is convex. In this case,* $p(t) = \nabla u \cdot t - u(t)$*, and* $x(t) = \nabla u(t)$*.*

Therefore, the allocation of a type t is $|\nabla u(t)|$, the $L1$ norm of the vector $\nabla u(t)$. Let W be the space of convex utility functions u. Let c be the cost of producing an item. Using the above lemma, we can reformulate the *rev* problem as follows:

$$\begin{aligned} \text{maximize} \quad & \int_T [\nabla u(t) \cdot t - u(t)] f(t) \,\mathrm{d}t - c\vec{1} \cdot \int_T \nabla u(t) f(t) \,\mathrm{d}t \\ s.t. \quad & u \in W \\ & \forall S \subseteq T, \int_S |\nabla u(t)| \,\mathrm{d}t \leq X(f(S)). \end{aligned}$$

For any $t \in T$, define the function $r_t : [0,1] \to T$ as follows:

$$r_t(x) = x \cdot t$$

Notice that $r(0) = 0$, and $r(1) = t$. We now use the gradient theorem and write

$$\forall t, u(t) - u(0) = \int_0^1 \nabla u(r(x)) \cdot r'(x) \,\mathrm{d}x.$$

In the optimal solution, $u(0) = 0$. Also, by definition of r, $r'(x) = t$. Therefore we have

$$u(t) = \int_0^1 \nabla u(x \cdot t) \cdot t \,\mathrm{d}x.$$

Using the above equation, we can rewrite the objective function as

$$\begin{aligned}&\int_T \nabla u(t)\cdot(t-c\vec{1}) - \int_0^1 \nabla u(x\cdot t)\cdot t \,\mathrm{d}x\,\mathrm{d}t\\ &= \int_T \nabla u(t)\cdot(t-c\vec{1})\,\mathrm{d}t - \int_0^1 \int_T \nabla u(x\cdot t)\cdot t\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

In the second term, change variables by defining $v = (v_1, v_2) = xt$. Notice that $t = v/x$, and $dv_i = xdt_i$ for any $1 \le i \le m$. Therefore $dv = dt/x^m$. Define T_x to be $t \in T$ such that $\max_{t_1,t_2} \le x$. The objective is

$$\begin{aligned}&\int_T \nabla u(t)\cdot(t-c\vec{1})\,\mathrm{d}t - \int_0^1 \int_{v\in T_x} \nabla u(v)\cdot(v/x)\cdot(1/x^m)\,\mathrm{d}v\,\mathrm{d}x\\
=\;& \int_T \nabla u(t)\cdot(t-c\vec{1})\,\mathrm{d}t - \int_{v\in T} \nabla u(v)\cdot v \int_{x=\max_i v_i}^1 1/x^{m+1}\,\mathrm{d}x\,\mathrm{d}v\\
=\;& \int_T \nabla u(t)\cdot(t-c\vec{1})\,\mathrm{d}t - \int_{v\in T} \nabla u(v)\cdot v\Big(\frac{1}{m\max^m v_i} - \frac{1}{m}\Big)\,\mathrm{d}v\\
=\;& \int_T \nabla u(t)\cdot\Big[t\Big(1+\frac{1}{m} - \frac{1}{m\max^m t_i}\Big) - c\vec{1}\Big]\,\mathrm{d}t\\
=\;& \int_T \nabla u(t)\cdot\Big[t\Big(\frac{m+1}{m} - \frac{1}{m\max^m t_i}\Big) - c\vec{1}\Big]\,\mathrm{d}t.\end{aligned}$$

We now show that relaxing the convexity constraint on u does not affect the optimum solution. Consider a type t such that $t_1 = \max_i t_i$ (the other cases are similar). Since $\nabla_1 u(t), \ldots, \nabla_m u(t)$ appear in the same set of constraints, we can assume that $\nabla_i u(t) = 0$ for all $i \neq 1$ in the optimum solution. We can therefore remove such variables. This converts the problem into a polymatroid optimization. Since $(\max_i t_i)(\frac{m+1}{m} - \frac{1}{m\max^m t_i})$ is non-decreasing in t and the the feasibility constraints are symmetric, we conclude that types are ordered by $\max_i t_i$. This implies that the optimum solution is as follows:

$$u(t) = \begin{cases} 0 & \max_i t_i \le \hat{t}_c \\ \int_{x=\hat{t}_c}^{\max_i t_i} X'(1-x^m)\,\mathrm{d}x & \max_i t_i > \hat{t}_c, \end{cases}$$

where $\hat{t}_c$ solves

$$t\Big(\frac{m+1}{m} - \frac{1}{mt^m}\Big) = c$$

Note, for example, that $\hat{t}_0 = \sqrt[m]{\frac{1}{m+1}}$. This function u specified above is convex and linear in X.

A.4 Proofs from Section 4.4.2

We now give a general procedure for implementing marginal revenue maximization with general agents. Consider an agent with interim allocation constraint $\hat{x}^{MR}$ (from marginal revenue maximization) and mix over $\hat{q}$-constrained lottery pricing with probability density given by $-\frac{\mathrm{d}}{\mathrm{d}\hat{q}}\hat{x}^{MR}(\hat{q})$ to get outcome rule $\tilde{w}^{MR}$. Recall each ex ante constrained lottery pricing is derived by optimizing revenue subject to a step function constraint on the allocation rule. The resulting outcome rule induces a normalized allocation rule that may not be a step function. Therefore, x^{MR} is dominated by but not necessarily equal to $\hat{x}^{MR}$. A natural approach to deriving an ordering on types and thus mapping types to quantiles would be to use the mapping given by Quant$(\cdot)$ for x^{MR}, where quantile of a type is the probability that a type drawn at random has a higher probability of service. This approach would result in allocation rule $\hat{x}^{MR}$ not the desired allocation rule x^{MR}. Instead the map from types to quantiles needs to be randomized to make it "worse" and equal to the desired allocation rule x^{MR}.

For uniform distributions over discrete type spaces, domination of allocation rules is equivalent to vector majorization. Hardy, Littlewood, and Pólya [30] show a transformation can be given between one and the other via a doubly stochastic matrix. This implies an algorithm for mapping types to quantiles based on solving a quadratic sized linear program: solve for the doubly stochastic matrix that gives x^{MR} from $\hat{x}^{MR}$, map an agent type to quantile via Quant$(\cdot)$ induced by x^{MR}, and randomly map this quantile to a distribution over quantiles by sampling according to the probabilities in the quantile's row in the matrix.

As we do not need to construct the matrix, but only to sample from a specific row there is a much simpler construction. This construction further generalizes to non-uniform distributions. Instead of permutation matrices this approach is based on applying a sequence of interval resamplings. An interval resampling is given by an interval $[a, b]$ and if the quantile is in $[a, b]$ it is resampled uniformly from $[a, b]$, otherwise it is unchanged. For allocation rule x, resampling quantiles from $[a, b]$ has the effect of replacing the interval of the cumulative allocation rule X with the line segment connecting $(a, X(a))$ to $(b, X(b))$. The construction below, for type spaces of size $m = |T|$, calculates the requisite sequence of at most m interval resamplings in linear time.

Definition A.4.1. For allocation constraint $\hat{x}$ and dominated allocation rule x satisfying $\hat{X}(1) = X(1)$ on m discrete types, the *interval resampling sequence construction* starts with $x^{(0)} = \hat{x}$ and calculates $x^{(j+1)}$ from $x^{(j)}$ while $x^{(j)} \neq x$ as follows.

(a) Find the highest quantile q where $x(q) \neq x^{(j)}(q)$.

(b) Let $q' > q$ be the quantile at which the line tangent to X at q with slope $x(q)$ crosses $X^{(j)}$.[2]

(c) The jth resampling interval is $[q, q']$.

(d) Let $x^{(j+1)}$ be $x^{(j)}$ averaged on $[q, q']$.

Proposition A.4.1. *The interval sampling sequence construction gives a sequence of at most m intervals such that the composition of $\hat{x}$ with the sequence of resamplings applied to* Quant$(\cdot)$ *is equal to x.*

Proof. The proof is by induction on j where the jth step assumes the first $j-1$ types, in order of Quant$(\cdot)$, satisfy $x^{(j-1)}(\text{Quant}(t)) = x(\text{Quant}(t))$. Consider step j. The assumption that $\hat{X}(1) = X(1)$ ensures that the intersection of the tangent happens at a $q' \leq 1$. The line segment connecting interval $[q, q']$ of $X^{(j)}$ has slope equal to $x(q)$, by definition.

[2]For discrete type, this intersection may happen at a quantile q' that does not correspond to the boundary between two types. When this happens split the type into two types each occurring with the same total probability and with the boundary between them at q'.

Therefore, the jth step in the construction leaves $x^{(j)}(\text{Quant}(t)) = x(\text{Quant}(t))$ for the jth type. The procedure is linear time as both $\hat{x}$ and x are, without loss of generality, piece-wise constant with m pieces, and in each step q and q' are increasing and at least one piece from $\hat{x}$ or x is processed. $\square$

The final ingredient in the construction of the marginal revenue mechanism for agents with general types is in converting the allocation rule back into an outcome rule. This can be done exactly as in Alaei et al. [1]: if an agent with type t is served by the allocation rule, sample from service outcomes of $\tilde{w}^{MR}(t)$, otherwise sample from non-service outcomes of $\tilde{w}^{MR}(t)$.

Definition A.4.2. The *marginal revenue mechanism* for general agents works as follows.

(a) Map reported types $\boldsymbol{t} = (t_1, \ldots, t_n)$ of agents to quantiles $\boldsymbol{q} = (q_1, \ldots, q_n)$ by, for each agent, composing the interval resampling transformation with $\text{Quant}(\cdot)$.

(b) Calculate the marginal revenue of each agent i as $R_i'(q_i)$.

(c) Calculate the set of agents to be served by marginal revenue maximization.

(d) Calculate outcomes for each agent i as:

- sample $w_i \sim \tilde{w}_i^{MR}(t_i)$ conditioned on $\text{Alloc}(w_i) = 1$ if i is to be served, or
- sample $w_i \sim \tilde{w}_i^{MR}(t_i)$ conditioned on $\text{Alloc}(w_i) = 0$ if i is not to be served.

Note that instead of calculating outcome rules by mixing over step mechanisms we could, from the allocation constraint $\hat{x}^{MR}$ for an agent, calculate the optimal mechanism subject to that constraint, i.e., with outcome rule $\text{Outcome}(\hat{x}^{MR})$ and revenue $\text{Rev}(\hat{x}^{MR})$. The construction above can be invoked with this outcome rule in place of $\tilde{w}^{MR}$ without modification; this change generally improves revenue.

The assumption in the interval resampling sequence construction that the allocation constraint $\hat{x}$ and the desired allocation x have cumulative allocations satisfying $\hat{X}(1) =$

$X(1)$ can be removed in downward closed settings. With this constraint removed the line tangent to X at q may be strictly below $\hat{X}$ at 1. If this happens we set $q' = 1$ and probabilistically reject a quantile falling in this interval. The probability of rejection is set so that the slope of the resampled allocation rule is $x(q)$.

Proposition A.4.2. *In downward closed settings, a normalized allocation rule x dominated by an allocation rule $\hat{x}$, where $\hat{X}(1) > X(1)$, can still be implemented by interval resamplings given in Definition A.4.1, and a marginal revenue mechanism for x can run similarly as in Definition A.4.2.*

We use $\hat{x} \geq x$ to refer to this weaker definition of dominance in downward closed settings.

A.5 Proofs for unit-demand approximation

Theorem 4.5.3 is a consequence of the two lemmas below and Proposition 4.5.2.

Lemma A.5.1. *Twice the optimal representative revenue is a linear upper bound on the optimal unit-demand revenue.*

Proof. Linearity follows simply from the revenue linearity of single-dimensional agents. Consider the distribution of the maximum virtual value (or zero if the maximum virtual value is negative) in the representative environment. Index this distribution by quantile as $\psi_{\max}(\hat{q})$. The optimal revenue for any allocation constraint $\hat{x}$ is $\mathbf{E}_{\hat{q}}[\psi_{\max}(\hat{q})\hat{x}(\hat{q})]$ which is linear in $\hat{x}$; this follows from the proof that the optimal revenue in single-dimensional environments is the virtual surplus maximizer.

We now show that twice the optimal representative revenue upper bounds the optimal unit-demand revenue. To do this we will give two auctions for the representative environment with the allocation constraint $\hat{x}$ and show that the sum of these auctions' revenue upper bounds the optimal unit-demand revenue for the same constraint. Of course, optimal representative revenue upper bounds each of these auctions revenue.

A mechanism for the unit-demand problem is simply a lottery pricing, i.e., it is a set of lotteries L with each $\ell \in L$ taking the form of $(p^\ell, \pi_1^\ell, \ldots, \pi_m^\ell)$ with $\sum_j \pi_j^\ell \leq 1$. The semantics of a lottery ℓ is that the agent pays the price p^ℓ and then is allocated an item j at random with probability π_j^ℓ; the semantics of the collection of lotteries L is that the agent, upon drawing her type from the distribution, chooses the lottery $\ell \in L$ that maximizes her utility (or none).

Given any collection of lotteries L that satisfies the allocation constraint $\hat{x}$ we define two auctions for the representative environment that have combined revenue at least that of the collection of lotteries in the unit-demand environment.

The L *mimicking auction* considers the profile of values $\boldsymbol{v} = (v_1, \ldots, v_m)$ of the representatives and the lottery $\ell \in L$ that would have been selected by the unit-demand agent with these values. It serves the representative j with the highest value with probability π_j^ℓ and charges her $p^\ell - \sum_{j' \neq j} \pi_{j'}^\ell v_{j'} + \mu(\boldsymbol{v}^{(2)})$ where $\mu(\boldsymbol{v}^{(2)})$ is the expected utility of the unit-demand agent with valuation profile $\boldsymbol{v}^{(2)}$ which is $\boldsymbol{v}$ with v_j replaced with $\max_{j' \neq j} v_{j'}$. Notice that the utility of the winning representative j in this auction is exactly the same as the unit-demand agent less an amount that is a function only of the values of the other representatives, $\boldsymbol{v}_{-j}$. As the utility of the unit-demand agent is monotone in her value for each item, the utility each representative has for winning is negative when she is not the highest valued representative and positive when she is. Therefore, this auction is incentive compatible, has revenue at least $p_j^\ell - \sum_{j' \neq j} \pi_{j'}^\ell v_{j'}$ on valuation profile $\boldsymbol{v}$ where j is the highest valued representative, and satisfies allocation constraint $\hat{x}$. For a given valuation profile, call the second term in the winning agent's payment the *deficit* of the L mimicking auction.

The motivation for the next auction is that we want to obtain back the deficit lost by the L mimicking auction. Notice that the procedure that charges the highest valued representative the second highest value and serves with probability $\sum_j \pi_j^\ell$ satisfies the allocation constraint $\hat{x}$ and more than balances the deficit; however, it may not be incentive compatible.

The *allocation constrained second-price auction* sells to the highest valued representative at the second highest representative's value so as to maximize revenue subject to the allocation constraint $\hat{x}$ that any representative is served. Consider the distribution of the second order statistic of values and let $v_{(2)}(q)$ be the value that the q quantile of this random variable takes on. The optimal revenue obtainable via a second price auction with allocation constraint $\hat{x}$ is $\mathbf{E}_q[v_{(2)}(q)\hat{x}(q)]$. To obtain this revenue, conditioning on the second highest value being v, with probability $\hat{x}(v_{(2)}^{-1}(v))$ we serve the highest valued representative and charge her v. This auction is incentive compatible and revenue optimal (in expectation) among all second-price procedures that meet the allocation constraint. Therefore, it more than covers the expected deficit of the L mimicking auction.

We have given two incentive compatible auctions for the representative environment with combined expected revenue exceeding the revenue of the lottery pricing L. Therefore, twice the optimal representative revenue is at least the optimal unit-demand revenue. $\square$

Lemma A.5.2. *The pseudo revenue curve $\tilde{R}(\cdot)$ from uniform virtual pricings for a unit-demand agent 2-approximates the optimal representative revenue curve (as a function of $\hat{q}$ for any $\hat{q}$-step constraint).*

Proof. Denote the optimal representative revenue for the $\hat{q}$-step constraint as a function of $\hat{q}$ by the revenue curve $\mathrm{ORR}(\hat{q})$. Consider the outcome of the optimal auction for the representative environment with ex ante service constraint $\hat{q}$. It sets a uniform virtual price (denoted $\psi(\hat{q})$) and serves the agent with the highest virtual value strictly bigger than $\psi(\hat{q})$ with probability one. If the probability that the largest virtual value is equal to $\psi(\hat{q})$ is strictly positive (which might happen if any virtual value function is constant on an interval, e.g., from ironing), it probabilistically accepts or rejects the maximum virtual value when it is equal to $\psi(\hat{q})$ so as to serve with the desired ex ante probability $\hat{q}$. The optimal representative revenue can thus be calculated and bounded as follows. Let $(\psi_1, \ldots, \psi_m)$ denote the profile of virtual values of the representatives.

$$
\begin{aligned}
\mathrm{ORR}(\hat{q}) &= \hat{q} \cdot \psi(\hat{q}) + \mathbf{E}\left[\max_i (\psi_i - \psi(\hat{q}))^+\right] \\
&\leq \hat{q} \cdot \psi(\hat{q}) + \sum\nolimits_i \mathbf{E}\left[(\psi_i - \psi(\hat{q}))^+\right].
\end{aligned}
$$

Above, the notation $(\psi_i - \psi(\hat{q}))^+$ is short-hand for $\max(0, \psi_i - \psi(\hat{q}))$.

Now we show a lower bound on $\tilde{R}(\hat{q})$ for $\hat{q}$ that does not require probabilistic acceptance in the optimal representative auction described above; denote by $Q \subset [0,1]$ all such quantiles. Let $\mathcal{E}_i$ denote the event that $\psi_j < \psi(\hat{q})$ for all $j \neq i$; our lower bound on the pseudo $\hat{q}$-constrained lottery pricing revenue will ignore contributions to the virtual surplus from the case that more than one representative has virtual value at least $\psi(\hat{q})$.

$$
\begin{aligned}
\tilde{R}(\hat{q}) &\geq \hat{q} \cdot \psi(\hat{q}) + \sum\nolimits_i \mathbf{E}\left[(\psi_i - \psi(\hat{q}))^+ \;\mid\; \mathcal{E}_i\right] \cdot \mathbf{Pr}\left[\mathcal{E}_i\right] \\
&\geq \hat{q} \cdot \psi(\hat{q}) + (1 - \hat{q}) \cdot \sum\nolimits_i \mathbf{E}\left[(\psi_i - \psi(\hat{q}))^+ \;\mid\; \mathcal{E}_i\right] \\
&= \hat{q} \cdot \psi(\hat{q}) + (1 - \hat{q}) \cdot \sum\nolimits_i \mathbf{E}\left[(\psi_i - \psi(\hat{q}))^+\right].
\end{aligned}
$$

The second inequality followed because $\mathbf{Pr}[\mathcal{E}_i]$, the probability of the event that $\psi_j < \psi(\hat{q})$ for all $j \neq i$ is not less than the probability that $\psi_j < \psi(\hat{q})$ for all j, which is $(1 - \hat{q})$. To extend this lower bound on $\tilde{R}(\hat{q})$ from $\hat{q} \in Q$ to all $\hat{q} \in [0,1]$, consider inserting a virtual value $\psi' = \psi(\hat{q}) + \epsilon$ with measure zero in the distribution. The $\hat{q}'$ that corresponds to serving this virtual value or higher has revenue bounded by the formula above but $\psi' \approx \psi(\hat{q})$. Keeping the virtual value constant and varying $\hat{q}$ in the formula interpolates a line between the two revenues. As the pseudo ex ante constrained lottery pricings are closed under convex combination, this line gives a lower bound on the pseudo $\hat{q}$-step mechanism. Therefore, the bound above on $\tilde{R}(\hat{q})$ holds for all $\hat{q}$.

To bound $\mathrm{ORR}(\hat{q})$ in terms of $\tilde{R}(\hat{q})$ we consider two cases. When $\hat{q} \leq 1/2$ these terms can be directly bounded as the first terms in both bounds are the same and the second terms are within a factor of two of each other (by assumption $1 - \hat{q} \geq 1/2$). When $\hat{q} \geq 1/2$ we can compare the bound on $\mathrm{ORR}(\hat{q} = 1)$ to the bound on $\tilde{R}(\hat{q} = 1/2)$; these bounds are within a factor of two of each other. Monotonicity (via downward closure) of $\mathrm{ORR}(\hat{q})$ and $\tilde{R}(\hat{q})$ then implies that they are within a factor of two for any $\hat{q} \in [1/2, 1]$. $\square$

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